THE TEACHING OF ARITHMETIC IN SCHOOLS

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A REPORT
PREPARED FOR THE
MATHEMATICAL ASSOCIATION

LONDON
G. BELL & SONS, LTD.

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The Boys' Schools Committee was composed as follows:

A. L. Atkin, District Inspector of Schools, L.C.C.
*H. F. Atwill, Stanley St. Elementary School, L.C.C.
F. C. Boon (*Secretary*, Dec. 1929 to Oct. 1930), Dulwich College.
*G. T. Clark, Loughborough Central School, L.C.C.
C. T. Daltry (*Secretary*, Oct. 1930 onwards), The Roan School, Greenwich.
W. C. Fletcher, late Chief Inspector of Secondary Schools, Board of Education.
E. H. Fryer, Charterhouse.
*A. S. Grant, 'Hillside,' Reigate.
*J. H. Hope, Ickwell Bury, Biggleswade, Beds.
W. Hope-Jones, Eton College.
A. B. Oldfield, The Grammar School, Pudsey.
A. Robson, The College, Marlborough.
R. V. H. Roseveare, The College, Cheltenham.
*A. W. Siddons, Harrow School.
H. P. Sparling, Rugby School.
C. O. Tuckey (*Chairman*), Charterhouse.
R. M. Wright, Winchester College.

* Co-opted members

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SINCE the *Arithmetic Report* was prepared the following articles have appeared in the *Mathematical Gazette*, volumes XVI-XXI.

XVI XVII	(1932) p. 262 (1933) p. 116	Accuracy of Figures	H. Berry
XVI XVII	(1932) p. 306 (1933) pp. 128, 326	Approximations	S. Inman and A. S. Percival
XVI	(1932) p. 320	The × sign	Saradakanta Ganguli
XVIII	(1934) p. 108	Slide-Rule, Logarithms, Indices	E. J. Atkinson
XVIII	(1934) p. 120	Logarithms	H. W. Oldham
XVIII	(1934) p. 192	Broadening the basis of study	H. M. Cook
XIX	(1935) p. 40	Inverted Practice	E. H. Neville
XIX	(1935) pp. 42, 218	The decimal point	R. F. Muirhead, J. W. Stewart
XIX	(1935) p. 213	Why use tables?	E. A. Guggenheim, E. H. Neville
XIX	(1935) p. 294	Calculation of Logarithms	F. C. Boon
XX	(1936) p. 324	Indices and Logarithms	E. V. Smith
XX XXI	(1936) p. 331 (1937) pp. 289, 290	Long Division for small divisors	A. A. Fletcher-Jones, C. D. Langton, and E. B. Escott
XXI	(1937) p. 117	Trial and Error and Approximations	A. C. Aitken

The Teaching of Arithmetic in Schools

§1. INTRODUCTION

§1·1. PURPOSE AND SCOPE OF REPORT.

1·11. This report is the first of a series of reports on the teaching of elementary mathematics which it is proposed to issue to bring up to date various reports of the Mathematical Association.

1·12. The present section discusses the teaching of arithmetic from what may be called the Preparatory School stage to the School Certificate stage. The Report on the Teaching of Mathematics in Preparatory Schools,* so far as it deals with arithmetic, has been consulted, but no effort has been made to avoid occasional repetition of important material. The two reports are complementary, but overlap.

1·13. This report includes sections on logarithms and graphs, which some schools may include in their algebra schedule, and a good deal of work on mensuration which many schools will include in their geometry schedule; this will largely depend on the text-books in use, and the committee does not wish to suggest that all these topics should be introduced into books on arithmetic, but they considered it desirable to consider the arithmetical aspect of these subjects in their arithmetic report.

§1·2. THE IMPORTANCE OF ARITHMETIC.

1·21. Up to the end of the last century, school mathematics for boys up to sixteen or seventeen generally consisted of arithmetic, algebra and geometry (Euclid) and there was very little correlation between the subjects. To-day, before the age of sixteen or seventeen the boy will have done a great deal of arithmetic in connection with algebra, geometry, mensuration, trigonometry and perhaps mechanics, so that the actual time needed for arithmetic as such should be considerably less than in former times. But this does not mean that arithmetic is less important than it used to be; it means that the applications of arithmetic are spread over a wider and more fruitful field. This very fact increases the actual importance of speed and accuracy in computation.

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* G. Bell and Sons, Ltd., 1928.

1·22. Moreover in teaching algebra there is a constant appeal to arithmetical methods: a boy acquires a feeling for handling letters in a particular process through studying the working of a similar exercise in integers. Again, though most boys will learn to reason correctly through working with geometrical diagrams, the duffer may fail to do so. Probably only through examples in arithmetic will he learn anything of correct mathematical reasoning. Arithmetic is often the only subject in which mastery is attained. Lastly, arithmetic lies much closer than other branches of school mathematics to the affairs of ordinary life. Good thorough teaching of arithmetic is vital. It should be remembered that a boy who takes School Certificate at sixteen to seventeen has learnt arithmetic for eight years, algebra and geometry for perhaps four, trigonometry, it is to be hoped, for perhaps two. More harm is done by teaching arithmetic badly for eight years than trigonometry badly for two.

§ 1·3. PLAN OF REPORT.

The report discusses the topics of school arithmetic, arranged only very roughly in an order in which they may be taught. Where the principles underlying a particular topic are clear and there is little difference of opinion as to the method by which they should be taught, the discussion is naturally brief. Usually a note added to some worked examples has seemed adequate; an appendix of worked examples has therefore been compiled to supplement those in the body of the report. But in other topics, e.g. the multiplication and division of decimals, discussion has been long, and widely differing opinions have been held. In discussing such topics, the report has preferred to state fully the contrasting views: this has involved discussion of principles and the inclusion of detailed instructions for teaching those principles.

§ 2. GENERAL REMARKS ON TEACHING ARITHMETIC

2·10. Teaching is an art in which skill can only be developed by patience, experience and observation. This report attempts to give advice as to sound methods and cautions against error. Neither the advice nor the cautions can be digested *en bloc*, except by those to whom they are already familiar and therefore perhaps unnecessary. To others, reference to that section of the report which touches the work they have in hand at the moment is likely to be more profitable than detailed study of the whole.

2·11. If a young teacher cannot digest too much at once, still less can his pupils. There is truth in an old proverb reversed, 'Cure is better than prevention.' That is, do not overload a lesson with cautions or minute directions ; make the general outline as clear as possible, but in detail leave boys to find their own way, and learn from their own mistakes.

2·12. In mathematical education the development of power is more important than the acquisition of knowledge ; hence it is necessary always to be very chary of giving help. One boy needs more than another, no doubt, but a great part of the skill of a teacher in his everyday work lies in knowing how much, or rather how little, is required ; too much assistance is as harmful as too little.

2·20. The same principle applies to the correction of boys' work : the ideal is to give the least amount of help which will enable the boy to put right what is wrong—it may be simply a word, 'that's wrong,—do it again' ; it may be 'the mistake is there, get it right' ; sometimes a fresh explanation may be required. A common method of dealing with errors in computation is to give the pupil 'another sum like it.' If the pupil is unable to track his error this defeats its purpose. In the correction of a solution containing several steps, all that should be imposed is the reworking or rewriting of the part of the solution which was inaccurate or defective.

2·21. Corrections must not be allowed to become burdensome either to the teacher or to the boys ; it is pedantic—and indeed impossible—to require that every sum attempted should be got right. And if there is wholesale failure or inaccuracy it probably means either that the work has been wrongly set or that there is something very wrong with the *morale* of the class, or the teacher, or both.

2·30. Much of the value of arithmetic is lost if it is not remembered that computation is an essential and important part of the subject. Any lack of respect for computation means a loss of educational value in accuracy, orderliness and quiet deliberation. Most teachers can recall pupils who set out all working as an integral part of their solution in a way that makes their work a delight to look at and whose accuracy is of the highest possible order. Bad habits and slovenliness acquired in arithmetic are fatal elsewhere, yet accuracy, neatness, form and discipline may be acquired here with the minimum of external aid from instruments or experiments.

2·31. In the early stages of arithmetic, the teacher should often give *viva voce* practice for a few minutes at a time in simple computation. Quickness and accuracy in addition, subtraction and the use of the multiplication table are only to be obtained by frequent practice. Therefore there should be short periods of steady drill directed mainly to that end.

This practice may be given by requiring the answers to ten rapid questions to be written down on slips of paper; but any practice is of little value unless it is systematic and regular. Such work gives a pupil confidence in his powers and has an obvious usefulness in everyday affairs.

2·32. Drill in computation may be made more attractive if boys are given the chance of setting their own questions. Usually boys will be found keen on making these quite difficult. This will be a safeguard against any temptation to choose or select sums which 'come out easily.' For instance, practice in making 'rough estimates' may be obtained in this way.

2·40. In written work, the importance of a clear and methodical arrangement should be insisted upon from the first. Sums may be roughly divided into two main classes, those in which the arithmetical computation is the main part of the solution, and those in which the development of the argument is more important, and computation is incidental to this. In writing out answers to the former class, the pupil should be taught to arrange his numerical work in the middle of the page. In the second class, which includes all questions of a 'problem' type, the middle of the page should be occupied by a series of sentences developing the argument, and the actual computation, which should be done neatly and should never be regarded as unimportant, should be placed below or at the side according to circumstances.

2·41. The computation should never be called 'rough work,' as doing so encourages the pupil to make it rough. Similarly, in geometry the term 'rough figures' should not be used. 'Side work,' or 'computation,' and 'freehand figures' are suggested as suitable alternatives.

2·42. In this connection it is important to notice that any laborious computation should be delayed to as late a stage in the working as possible. The pupil may appreciate this point better if he is told that no work should be done that can be avoided, and that in the end a long piece of multiplication or division, which is apparently impending, may prove to be unnecessary when factors are 'cancelled.'

2·43. Again, two divisions sometimes become one multiplication. But any simplification that can be done easily should be performed at once. Cancellation should be done neatly. The ideal policy is that numbers should not be crossed out at all, the whole expression being rewritten; but it is a policy which few teachers adopt themselves, and is perhaps not to be urged strongly, provided the work is kept neat and legible. Symbols of abbreviation, such as π and $\sqrt{2}$, should be retained as long as possible.

2·44. A word may be said here about two complementary dangers. Some teachers will expect processes to be done mentally which, with their own constant practice, have become both simple and safe to themselves, but which would tend to inaccuracy in the pupil and which, indeed, would be beyond the power of an adult not engaged in teaching arithmetic.

The other danger is illustrated by the boy who uses logarithms, without thinking, when simple division may be all that is needed; or uses long division to divide by 20.

2·50. The wording of the solution of a question should be in simple, precise English, and its meaning should be intelligible independently of the wording of the question. The final answer should be clearly stated; in a computation sum it will be a number or numbers and their units; in a problem sum it will be a definite statement, in a complete sentence, of the result attained. [See the worked examples on pp. 77 to 82.] The last and least essential word to be written is the word 'Answer.' This word, if it is thought necessary, may conveniently take the exact position of the Q.E.D. in geometry.

2·51. The foregoing principles of arrangement and style apply equally strongly to examples in science and applied mathematics, and science masters should be urged to insist upon a high standard in these respects.

2·6. All teaching of general principles should begin from simple concrete examples, and the preliminary work leading up to a broad generalisation should be mainly oral. The final generalisation into a rule-of-thumb should be delayed until it is almost spontaneously suggested by the pupils themselves. It is vitally important to give a clear justification for any rule-of-thumb; some pupils will understand it, a few will remember it, but all should realise that there is a justification and that it is logic and not magic.

On the other hand, it is most important that this final crystallisation of a fluid method into a hard-and-fast rule should be well defined. The pupil must know and use such rules-of-thumb, and the habit of using them must be developed. Mathematics essentially aims at an economy of thought, and the more operations a boy can perform without having to stop and think, the more he will learn about the subject. 'Civilisation,' as Whitehead says, 'advances by extending the number of important operations we can perform without thinking about them.'

Whenever a new principle or a new application of arithmetic is being taught, the first examples should always be of a simple nature and should not involve heavy numerical work. In general, whenever the argument is difficult, the numerical work should be easy, and *vice versa*.

2·7. The exposition of new matter should seldom be lengthy. Boys, especially young boys, flag and lose interest unless they are themselves doing something more active than listening. Hence exposition should be given in short doses interspersed with the working of examples.

2·8. The nature of the examples used will depend upon the type of school and boy concerned. Every effort should be made to bring the teaching of arithmetic in the class-room into close relation with the life of the boy outside it. Many examples for young boys can be taken from the games in which they are most interested; later, when vocational interest is more clearly defined, they can be taken from science, commerce and finance.

2·9 The opportunity should be taken at suitable points in the arithmetic course to introduce the use of letters for numbers and so prepare the way for algebra.

Most of the rules are much shorter in algebraic symbols and their statement in symbols forms a final generalisation of great value in making clear the processes to be used as well as in leading up to future work.

Instances of such rules are $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}$, $A=l\times b$ for area of a rectangle, $A=2(l+b)h$ for walls of a room and, later on, the formulae for simple and compound interest.

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§ 3. STYLE

3·1. Feeling for style and accuracy of expression, and for form in the presentment of work, is of gradual growth, like all mental attainment. A teacher has to consider this dilemma : should he require the accurate use of symbols and words from the start, so as to lay the foundations of good habits and prevent bad ones from being acquired, or will such a course tend to make the work seem artificial to the boy, if required of him before he reaches the age and stage of development which appreciates and feels the need of such style ? Children use a small vocabulary for ordinary talk, not only through ignorance of more words but owing to their small range of experience and to their ignorance of finer shades of meaning. Words are constantly used loosely ; and this is quite natural. In teaching mathematics, there is a danger in demanding too much in the way of style if the demand is made too soon. Good judgment alone can determine how much accuracy of expression should be required at any given age.

3·2. The matter can only be decided in an individual case by the teacher keeping his hand on the pulse of the class ; their interest will be found to flag when there is too great insistence on what is, to the immature mind, merely detail.

3·3. A good style in the presentation of mathematical work is a valuable aid to accuracy and leads to economy of thought and time. Good style consists in telling the truth and telling it as clearly as possible : it is often the index of ability, but any boy who thoroughly understands his solution of a problem need take but little extra time to present it in a reasonable form, i.e. to say what he means in each line of the solution. Persistent bad style is often the result of failure to understand, but apart from this it must not be tolerated : for a stage is reached, and quickly reached by weaker pupils, when the lack of a good style constitutes a definite handicap to progress.

3·4. A style from which the pupils will learn much is that of the teacher. When the blackboard is used for what are claimed to be illustrations of methods of working or for the display of complete solutions, these should be well presented.

3·5. It should be impressed on pupils that a well-presented solution reads as connected English, in which verbs are essential, unless the quantities that occur are tabulated. But in mathematical work the use of recognised symbols and notation is permitted, and so is the intelligent abbreviation of some of the words that occur. The careful presentation of a solution is therefore an exercise in the art of writing English,* and it is not unprofitable for an occasional lesson to be specially directed to this aspect of mathematical work. (A problem should be selected of sufficient difficulty to hold the attention of the class, the solution should be reached by oral discussion, the essential steps being stated orally, and finally the whole solution should be built up on the blackboard.) Oral teaching should be directed partly to the formation of mathematical style, not exclusively of habits of mathematical thought, though this aim also enters in some degree into a discussion of the kind suggested.

3·6. Some cautions must here be entered for the teacher.

The indifferent style which is sometimes adopted by the abler students may be due to their being given work which is too easy for them. A good style is more likely to be fostered or maintained by work which is not devoid of difficulty and is therefore worth doing well. Again, teachers, usually those with some experience, may too soon require perfection of style. This has the effect of substituting the mental operations of the teacher for those of the pupil. A mean has to be observed between the toleration of slipshod work and a requirement of precision which amounts to a dictation of the form in which the problem should be set out. To each age correspond its degree of accuracy, its vocabulary, and its standard of exactness or elasticity of thought and expression. Every traveller on the Continent is aware of the limitations imposed on his style in conversation by a limited vocabulary.

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* The pamphlet, *Some Suggestions for the Teaching of English in Secondary Schools in England* (H.M.S.O., 6d.), especially Appendix I, 'English and Mathematics,' may be consulted with advantage in this connection.

§ 4. ALTERATION OF HABITS

4·1. No attempt should be made to alter those habits which involve a definite process of thought, as opposed to those which concern merely mechanical figure work. The method of subtraction is an example of the former ; provided a boy can subtract correctly, it does not matter whether he thinks of it as taking away or as complementary addition.

4·2. On the other hand, in multiplying, boys should be taught to begin with the leading digit of the multiplier, whatever they have learnt before, or in dividing, to put the figures of the quotient over the dividend. These and other methods of mechanical work are worth altering, if they affect later work.

4·3. Habits which tend to an economy of thought should be encouraged. The alteration of some, which are perhaps inevitably introduced in the very early stage, forms a definite object in later teaching. Such habits as that of naming each figure in addition (' 7 and 5—12—and 6—18 ' for ' 7, 12, 18 ') should be corrected by frequent drill. In this connection also the form of words adopted by the teacher in working examples on the blackboard is of the utmost importance.

§ 5. CHECKS

5·1. Everyone who takes any real interest in his arithmetical or mathematical work takes pains to secure accuracy both of thought and mechanical execution. One of the distinguishing features of the subject, indeed, is the possibility of obtaining substantial certainty—and if this is neglected much of its general educational value is lost. Further, in guarding against mistakes and in locating them when error is manifest, the matter in hand frequently becomes more fully understood—improvements in method suggest themselves, skill and insight are developed.

5·2. Boys, of course, are commonly unwilling to take the necessary trouble—first from sheer indifference: why bother to look at a thing a second time to see if it could be improved, or even to see if the answer is reasonable?—perhaps also from greed for marks, it may pay better to do half a dozen sums and take one's chance than to do three and take pains; and the main business of the teacher is to try to awaken this zeal for accuracy—difficult to secure, because it is moral as well as intellectual.

5·3. It is necessary to keep this general principle clearly in mind or the mere insistence on 'checks,' whether 'rough' or otherwise, may fail of its purpose even in the limited sphere in which it is exercised. Thus boys who are habituated to writing out checks of solutions of equations will often quite contentedly leave side by side a 'solution' and a check which are not in accord; or will get a rough estimate to a sum in decimals or a logarithmic evaluation and entirely disregard contradiction between the estimate and their actual answer.

5·4. The educational process is only, even relatively, complete when a boy for his own satisfaction habitually applies such checks, tests, or reconsideration as shall make him fairly certain that his work is correct; and unless he is at least on the way to this frame of mind, insistence on his writing out such checks is apt to be futile. If the spirit—the desire for accuracy—is not awakened, a boy comes to regard a check as merely part of a sum—to be done and left without consideration. No progress can come that way.

5·50. The pupil who forms the habit of making rough preliminary estimates reaps real benefit in other ways than that of finding checks to his answers.

5·51. In dealing with questions in which the numbers are not simple and the argument is intricate, he will find it easier to get a line of thought if he does a preliminary sum in which the numerical values of the data are simplified.

5·52. Outside the class-room he will be able to make a usefully approximate estimate when an immediate solution of an everyday problem is desirable; e.g. the gallon-capacity of a cistern, the amount of money he is spending when taking a holiday on the Continent, etc.

5·60. Accuracy in the four fundamental rules is vital. The habit of checking addition by adding up and down should be steadily inculcated. It is hardly possible to enforce it, in the sense, that is, of requiring written evidence that it has been done—if a boy is determined to shirk it he can; but teachers should do all they can to secure it.

5·61. It is worth while to check multiplication by 'casting out the nines'; here it is possible to secure written evidence, but it is not suggested that this is desirable except from those who are suspected of shirking. Even with them, unless a change of spirit can be effected—unless a boy comes to use it of his own free will—little can be gained.

5·62. Subtractions ought perhaps to be checked by addition, but here again reliance has to be placed on the boy's honesty.

5·63. Division does not lend itself to any convenient check—it would be wasteful to check every division by multiplication. A good worker will devise his own partial checks—especially comparison of the various multiples of the divisor. But this requires more intelligence than is generally available.

5·7. In the multiplication and division of decimals it has become customary to require, as a regular part of the performance of the operation, the presentation in writing of the rough estimate which applies solely to the position of the decimal point. And with the poorer workers the result is not wholly satisfactory. To many, the rough estimate remains an isolated part of the work, and they fail to apply it to control the answer. That is to say, a process which applied intelligently and deliberately is a valuable help becomes merely an additional source of error and confusion to the unintelligent and the lazy. Possibly the extra burden is an actual disadvantage to them, distracting their limited intelligence from proper attention to the rule.

5·8. The true purpose of a check is to detect mere slips—not to be a necessary part of the rule itself, and the working should not be more complicated than the original working. If the rule itself is so tricky as to require the check some simpler rule should take its place. With the better workers the estimate probably soon comes to replace the rule, i.e. they fix the decimal point by the exercise of intelligence ; the weak need first and foremost a firm grasp of a clear rule.

5·9. Apart from the actual checking, boys need to be trained to consider whether or not their answer is reasonable—not, for instance, to leave the cost of fencing half an acre as £17,137 10s. 4½d. approx., or the result of investing £120 yearly at 5 % for 10 years as £223,741,935,483,870,967,741 (approx.) without comment, as was done in a recent School Certificate Examination. 'Common sense' needs acquiring, or at least developing, and constant attention to it is necessary.

§ 6. ARRANGEMENT OF WORK: SUMS AND PROBLEMS

6·1. The questions set in elementary *algebra* either fall clearly into a group known as problems, or they do not. There is seldom any question of putting 'side work' in the margin in either case. A large number of questions in arithmetic fall into a group corresponding to that of the algebra 'problem,' and, in these cases, it is recommended that any operation (such as long multiplication, division, square root, logarithms, etc.) which would disturb the sequence of the argument, should be performed on the right-hand side of the page. It is necessary to emphasize that such work should be 'rough' neither in accuracy, writing, nor arrangement. It may be added that the figures in such work should be of the same size as those used in the body of the work, and that no effort should be made to confine them to any small margin that may be ruled off on the paper.

6·2. Arithmetical questions not falling into this group, which we may call 'sums' by way of distinction, will mainly be those set in the course of learning an operation or for practice in computation. These should be worked wholly in the middle of the page. Such operations cannot wholly form part of a sentence, and any tendency to prefix the sign = should at once be checked. The abuse of this sign occurs also as a link with the word 'answer.'

6·3. The following are types of questions which fall into these two categories, but it should be understood that they are given here rather to make clear the distinction than to emphasize the position of any particular type.

Sums.—Multiplication and Division, whether Simple or Compound, Practice, Fractions, Square Root, Compound Interest.

Problems.—Unitary Method, Percentage, Proportion, Mensuration.

§ 7. SOME BRIEF REMARKS ON THE FIRST FOUR RULES

7·1. Practice should be given in adding small numbers, which are not necessarily arranged under one another.

7·2. Boys should be made familiar with the complementary nature of addition and subtraction, and of multiplication and division; 'What do you multiply 9 by to get 72?' should be asked as well as 'What is 72 divided by 9?'

7·3. Most boys have learnt to subtract before they come to school, and if a method is thoroughly known and the process has become mechanical, it is not advisable to try to alter it.

The 'shop' method is recommended.

EXAMPLE.	305	Say 8 (and 7 make) 15,
	148	5 (and 5 make) 10,
	157	2 (and 1 make) 3,

instead of '8 from 15 leaves 7,' etc.

In any case the method of 'equal additions' is to be preferred to 'decomposition.' By the former method the process in the above example would be '8 from 15 leaves 7, 5 from 10 leaves 5 (*not* 4 from 9),' etc.

7·4. The two aspects of division should be impressed on the pupil: e.g. he should be asked not only the question 'if 12 pence are divided into three parts, how much?'; but also, 'if 12 pence are divided into shares of three pence, how many?'

7·5. Boys should not be taught or allowed to read '12÷3' as '3 divided into 12,' a very usual practice which confuses the process with that of '3 divided into 12 parts,' a quite different matter. The correct reading of '12÷3' is '12 divided by 3.'

7·6. In most cases long division is to be preferred to division by factors, until decimals have to be dealt with.

7·7. The 'Italian' method of long division in which remainders alone are written down is not recommended.

7·8. H.C.F. should not be taught as a special rule, and the method of finding it by alternate division should be postponed until the theory can be understood.

7·9. The aim throughout this early work should be to get more of the simple processes done mentally. Boys should be progressively discouraged from writing down the divisor and drawing lines for short division, in working e.g. $118 \div 2$ or $\frac{245}{7}$.

§ 8. LONG MULTIPLICATION OF MONEY

8·10. Many methods are in use: none of them is easy. Fatal slips will occur in the work of usually reliable boys. Accuracy, with rapidity, is hard to attain.

8·11. Two methods are familiar and traditional—the 'Ten-ten' and 'Practice.' Each is an indirect or 'piecemeal' method as it has been called. A third is apparently more recent; it may be called the 'wholesale' method, since there is no preliminary working, and the sum is worked directly. It is finding increasing favour.

8·12. It should be noted that Long Multiplication can often be avoided by using factors.

8·20. *The 'Ten-ten' method.*

E.g. £87 16s. $9\frac{1}{4}$d. × 345 is found thus:

	£	s.	d.
(1)	87	16	$9\frac{1}{4}$
			10
(2)	878	7	$8\frac{1}{2}$
			10
(3)	8783	17	1

£	s.	d.	
439	3	$10\frac{1}{4}$	
3513	10	10	
26351	11	3	
30304	5	$11\frac{1}{4}$	*Answer.*

Regarding 345 as $5 + (4 \times 10) + (3 \times 100)$, we multiply the sums of money in rows (1), (2), (3) by 5, 4, and 3 respectively, obtaining the three rows on the right: and add.

8·21. This method has the advantage that multiplication by 10 is an easy operation. But with a multiplier running into thousands the preliminary forming of the rows is lengthy: and any mistake in the forming of the rows leads to an answer completely wrong.

8·22. Also the mental reduction by division by 12 and 20 alternating with the multiplication by 10 produces continual slips.

8·30. *The 'Wholesale' or 'Top Line' Method.*

This uses the figures in the top-line as multipliers: e.g.

£	s.	d.	f.
87	16	9	1
			345
289	265	86	4) 345
2415	3450	3105	86 + 1
27600	2070	12) 3191	
30304	20) 5785	265 + 11	
	289 + 5		

£30,304 5s. $11\frac{1}{4}$d. *Answer.*

8·31. The setting out may be vertical, thus :

$$\begin{array}{rrrrl}
\text{£} & \text{s.} & \text{d.} & \text{f.} & \\
87 & 16 & 9 & 1 & \\
 & & & 345 & \\
\cline{4-4}
 & & 4\,) & 345 & \\
\cline{4-4}
 & & & 86 & +1\text{f.} \\
 & & & 3105 & \\
\cline{4-4}
 & & 12\,) & 3191 & \\
\cline{4-4}
 & & & 265 & +11\text{d.} \\
 & & & 3450 & \\
 & & & 2070 & \\
\cline{4-4}
 & & 20\,) & 5785 & \\
\cline{4-4}
 & & & 289 & +5\text{s.} \\
 & & & 2415 & \\
 & & & 27600 & \\
 & & & \text{£}30304 & \\
\cline{4-4}
\end{array}$$

£30,304 5s. 11¼d. *Answer.*

8·32. The horizontal setting out can be abbreviated by omitting the divisors 4, 12, and 20, and writing the remainder in a space provided for the final product as shown :

£	s.	d.	f.	
87	16	9	1	
			345	
£30304	5	11	1	*Answer.*
289	265	86	345	
2415	3450	3105	86	
27600	2070	3191		
	5785	265		
	289			

8·33. Of this method it may be noted that although preliminary drill is needed in the wide-spacing and use of the figures in the top-line as multipliers, on the other hand each monetary unit is disposed of in turn, which is a definite convenience.

8·40. '*Practice.*'

By breaking up the sum of money into a set of aliquot parts (i.e. fractions whose numerators are unity) of £1, the multiplication is replaced by the simpler process of division.

8·41. Thus, using one set of aliquot parts, £87 16s. 9¼d. × 345 may be set out :

	£	s.	d.	
£1	345	0	0	
£80	27600	0	0	
£7	2415	0	0	
10s.	172	10	0	
5s.	86	5	0	
1s.	17	5	0	
6d.	8	12	6	
3d.	4	6	3	
¼d.		7	2¼	
£87 16 9¼	£30304	5	11¼	*Answer.*

8·42. Of this method it may be said :

(1) that it is elegant with a conveniently divisible sum of money, and often needs little more than repeated division by 2—the easiest arithmetical operation—e.g. for £1 18s. 9d. × 345 ;

(2) that it is almost essential in such a sum as : Find the cost of 3 tons 15 cwt. 1 qr. at £4 6s. 5d. per ton ;

(3) that the principle of the method is one of the most important in elementary calculation, and one which may be applied very widely, e.g. to such questions as

Find 7¼ % of £171 5s. 6d.,

Find the tax on £572 at 6s. 6d. in the £,

Given 1 dollar=4s. 1.3d., change 7513 dollars 35 cents to English money,

—indeed theoretically to any multiplication.

(4) Unless there is preliminary drill in the choosing of aliquot parts, the duller boy increases his chance of error by choosing too many parts.

(5) It always leaves something to the individual intelligence : and is not therefore easily made mechanical.

8·5. The Committee recommend that either the 'Ten-ten' or the 'Wholesale' method should be learnt at the '4 Rules' stage : and feel that teachers should test for themselves the latter.

The 'Practice' method must certainly be taught, but at a later stage.

§ 9. FRACTIONS

9·1. Easy fractions should be taught before decimals. Fractions such as $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{26}$, $\frac{1}{78}$, etc., appear in the papyrus which Ahmes copied *circ.* 1700 B.C., and his methods were still in vogue 2000 years later. Rules for handling fractions in general do not appear till about 1000 A.D. This slow growth may serve as a caution against hurrying the introduction of these rules to-day.

9·2. Concrete examples, such as fractions of an hour or lengths on a ruler, should come first. A boy should also understand that a fraction is the result of either of two processes. He can cut up a pudding into five equal parts, of which he can subsequently take three helpings ; he can also divide, say, 3 inches into 5 parts. At the same time, a boy's idea of number is extending beyond that of a mere series of integers.

9·3. As a preliminary to the addition of fractions, the comparison of small fractions is a useful exercise. In addition of fractions, stress should be laid on the principle involved in bringing fractions to a common denominator, viz. that a fraction is unaltered if its numerator and denominator are multiplied by the same number. At first fractions should be written with the same denominator separately before the fractions are combined. In the early stages it is also helpful to write the denominator as a word, e.g. '3 fifths.

9·4. Mixed numbers should not be reduced to improper fractions in addition and subtraction.

9·50. There is one point in regard to 'Fractions' to which teachers need to pay special attention, not that they may discourse learnedly to their classes, but that they may avoid teaching what is false—this is, multiplication by a fraction. According to the primitive definition, $a \times 5$ is an abbreviation for $a+a+a+a+a$, and the definition is nonsense unless the multiplier is an integer. Nothing can be *proved* about $N \times \frac{3}{4}$ until a meaning has been given.* The question is, what meaning, if any, will be useful and at the same time will not lead to contradictions. The difficulty is sometimes not clearly realised, because in the other operations—addition and subtraction, and in part in division—it can be got round ; so that the assumption is natural that it can be got round here also, i.e. that a rule for multiplication can be *proved* ; this is false.

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* The sign × must be read 'multiplied by,' not 'times.'

9·51. To see this clearly, consider in outline the whole treatment of fractions. To explain what a fraction is, we take something—a line, a rectangle, a heap of sand, a sum of money or a weight—and explain what $\frac{3}{4}$ of that thing means. And we then prove easily enough that $\frac{3}{4}$ of that thing equals $\frac{6}{8}$ of it, and so on.

9·52. We work in fact in concrete, not abstract fractions, and only when the teaching has gone home do we drop the reference to the thing divided and say $\frac{2}{3}+\frac{1}{4}=\frac{11}{12}$, instead of $\frac{2}{3}$ of an inch $+\frac{1}{4}$ of an inch $=\frac{11}{12}$ of an inch. On this basis we *prove* the rules for equality, for addition and subtraction.

9·53. This is right and proper for school work and is precisely analogous to the teaching of addition or subtraction to small children ; they begin, say, with 2 balls + 3 balls = 5 balls, and gradually and without overt notice make the transition to the abstract, $2+3=5$.

9·54. In each case the *proof* depends on the numbers or fractions being concrete ; a mathematician who is not content with this and wants to deal with abstract numbers and abstract fractions has to adopt quite a different procedure ; and his methods are not food for babes.

In multiplication no such resource is available : for a multiplier in the primitive sense is necessarily abstract as well as integral. There is no difficulty in multiplying or dividing a fraction by an integer ; these points will in effect have been dealt with already and, if not, the same treatment is available. There are various ways of taking the final and critical step ; perhaps the simplest is to *define* $N\times\frac{3}{4}$ as meaning $\frac{3}{4}$ of N, which is already understood—and to show by various examples that this is useful and sensible.

9·55. Thus to find out the cost of $5\frac{3}{4}$ yards of cloth at a given price involves, according to primitive methods, two separate sums, (*a*) 5 yards at so and so—an ordinary piece of multiplication, and (*b*) $\frac{3}{4}$ of a yard costs $\frac{3}{4}$ of so and so ; the definition adopted enables us to reduce the whole to one statement : multiply the price by $5\frac{3}{4}$. It follows that to multiply by $\frac{3}{4}$ we must multiply by 3 and divide by 4, i.e. multiplication by a proper fraction decreases the multiplicand—a point that often causes perplexity because of the natural association of the word "multiplication" with increase.

9·60. Division is the inverse of multiplication, i.e. if $N\times\frac{3}{4}=P$, then $P\div\frac{3}{4}=N$; i.e. N is derived from P by division by 3 and multiplication by 4, or shortly, 'to divide by a fraction turn it upside down and multiply.'

9·61. There are two methods of introducing division by a fraction

$$\text{(i) } \frac{\frac{3}{5}}{\frac{2}{7}}=\frac{\frac{3}{5}\times 7\times 5}{\frac{2}{7}\times 7\times 5}=\frac{3\times 7}{2\times 5}=\frac{21}{10};$$

$$\text{(ii) } \frac{\frac{3}{5}}{\frac{2}{7}}=\frac{\frac{21}{35}}{\frac{10}{35}}=\frac{21}{10}.$$

9·62. Of these, (i) seems to be the simpler in thought (it is merely an application of the principle mentioned in Paragraph 9·3 above). As the boy finally acquires the rule 'Turn the denominator upside down and multiply' the brevity of (ii) is no permanent gain.

9·63. There is a difference of opinion among teachers as to the wisdom of allowing the numerator and denominator to be scratched through when 'cancelling' is done. The policy that the whole expression should be rewritten, none of the numbers being crossed out, is an ideal one ; few teachers adopt it themselves, and it cannot be urged strongly, provided that the work is kept neat and legible. Mistakes in accuracy are frequently caused by small or carelessly written figures during this operation. Plenty of space is required above and below.

9·64. In no event should a boy be allowed to cross out a number by drawing a separate line through each digit. Difficulties of printing lead to numbers being crossed out in this way in the examples shown in some books.

9·7. At the stage at which fractions are usually learnt, boys have difficulty in dealing with numbers written as the product of two or more factors ; they do not at first see that, e.g. 4×35 divided by 7 is 4×5. In multiplication of fractions this difficulty occurs whether numbers are crossed out or the whole expression rewritten. It is necessary to direct some teaching to this point, as fuller understanding should tend to remove future confusion, which is particularly obvious in algebra.

9·8. A few points deserve notice.

1. The teacher should lead up to the rules by discussion and example, and, if possible, get them suggested by the class. Once this stage is reached, the class should be given a clearly defined statement of each rule, enforced with ample drill in written examples. If a boy has to reduce $\frac{\frac{4}{9}}{\frac{3}{4}}$, it is neither necessary nor desirable that he should think of the problem as that of finding out how many times $\frac{3}{4}$ is contained in $\frac{4}{9}$. He should think of it as a mechanical operation.

2. The lowest common denominator should be found by putting the various denominators in their prime factors, not by any formal method. The common denominator need not be the lowest.

3. Fractions of an unusual kind should not be set in the early stages; numbers of one digit are, in general, sufficient. Complicated fractions, as such, involving numbers of more than two digits, serve little purpose at any stage during the school course. There are better ways of spending the time.

§ 10. MULTIPLICATION AND DIVISION OF DECIMALS

10·10. In view of the variety of available methods explained below, uniformity of practice in all the schools in the country cannot be expected; and since boys will often be changing their schools at an age when they have recently learnt to work with decimals, it will often happen that a teacher is faced with pupils who have learnt or partially learnt different methods.

10·11. In order that a boy should not be confused by learning two different methods or even slight modifications of one method, it is highly desirable that, in the first term in which he does multiplication or division of decimals, he should have enough drill in them for the methods to be finally established and practically mechanical. The same principle holds good in other parts of the subject when alternative methods are liable to lead to confusion.

10·12. It is essential that a boy who has thoroughly learnt and mastered some method should not be forced to change it simply in order to secure uniformity in a school. Apart from this, it is desirable that within the same school the same methods should be used.

In what follows, various methods are explained and illustrated; then their advantages and disadvantages are pointed out; and finally the recommendations of the Committee are made.

§ 10·2. MULTIPLICATION.

10·21. *Method I.*

To multiply 23·612 by 42·1, multiply as if these were whole numbers and determine the position of the decimal point in the answer by dividing by the requisite power of 10, i.e. by counting the total number of decimal places.

$$\begin{array}{r} 23612 \\ 421 \\ \hline 94448 \\ 47224 \\ 23612 \\ \hline 9940652 \\ \hline \end{array}$$

The decimal point can be placed, if wanted, in each partial product *after* it has been placed in the answer.

994·0652 *Ans.*

To find $(\cdot 02)^3$. This has 6 decimal places, of which the last is 8. The answer is ·000008.

Similarly $\cdot 04 \times \cdot 003 = \cdot 00012$.

10·22. *Method II.*

Convert the multiplier to Standard Form, i.e. a number lying between 1 and 10. To do this, divide it by 10 and, to compensate, multiply the multiplicand by 10.

$$23{\cdot}612 \times 42{\cdot}1 = 236{\cdot}12 \times 4{\cdot}21.$$

$$\begin{array}{l} 236{\cdot}12 \\ 4{\cdot}21 \\ \hline 944{\cdot}48 \\ 47{\cdot}224 \\ 2{\cdot}3612 \\ \hline 994{\cdot}0652 \\ \hline \end{array}$$

10·23. *Method III.* Set down thus ('Tracking the unit' method):

$$\begin{array}{l} 23{\cdot}612 \\ 42{\cdot}1 \\ \hline 944{\cdot}48 \\ 47{\cdot}224 \\ 2{\cdot}3612 \\ \hline 994{\cdot}0652 \\ \hline \end{array}$$

As the multiplying digit 4 stands 1 place to the left of the units figure, the digits of the corresponding partial product move 1 place to the left.

10·241. *Method IV.* (*a*).

Set down thus:

$$\begin{array}{l} 23{\cdot}612 \\ 42{\cdot}1 \\ \hline 944{\cdot}48 \\ 47{\cdot}224 \\ 2{\cdot}3612 \\ \hline 994{\cdot}0652 \\ \hline \end{array}$$

The *units* digit of the multiplier is placed under the right-hand digit of the multiplicand, and in every partial product the first figure obtained goes under the multiplying digit.

10·242. *Method IV.* (*b*).

$$\begin{array}{l} 23{\cdot}6\;12 \\ 4\;2{\cdot}1 \\ \hline 94\;4{\cdot}48 \\ 4\;7{\cdot}224 \\ 2{\cdot}3612 \\ \hline 99\;4{\cdot}0652 \\ \hline \end{array}$$

The first significant figure of the multiplier is placed under the units digit of the multiplicand; the first figure obtained in the first partial product goes under the last figure of the multiplicand, and the decimal point under that of the multiplier.

10·25. *Remarks on the above Methods.*

10·251. I. (i) This method is the easiest to explain for it only uses the definition of a decimal; ·612 is $\frac{612}{1000}$.

In $\frac{23612}{1000} \times \frac{421}{10}$ the denominator will have 4 zeros; hence 4 decimal places.

(ii) If the multiplication is to be worked in full there is little doubt that this method is the most accurate. It has been described as 'very safe, very mechanical, and easily taught by the slovenly teacher.'

(iii) It is the *only* satisfactory method for examples of the type $(\cdot 02)^3$.

(iv) Its sole defect is that, the decimal point not being placed in the partial products, the method requires supplementing for contracted work.

10·252. II. (i) This method has been warmly recommended by the Mathematical Association and is widely used. It is only fair to say that its adoption has not had all the good effects hoped for.

(ii) The preliminary rearrangement of the decimal points amongst the figures leads to many mistakes.

(iii) The method has the advantage of leading up to work with logarithms. Boys who have learnt Standard Form either here or *in connection with division* find the introduction to logarithms made definitely easier for them.

(iv) It is suitable for contracted work.

10·253. III. (i) This method has the advantage, which it shares with (I), that there is no preliminary rearrangement of the decimal points amongst the figures; and the advantage, which it shares with (II), that the decimal point can be placed in the partial products.

(ii) It has the disadvantage that it is more difficult for such an example as ·272 × ·00058.

10·254. IV. (*a*) and (*b*).

(i) These have the same advantages as (III), and provide an automatic rule for placing the decimal point.

(ii) Their first disadvantage is that, owing to the two positions of the decimal point, unusually careful spacing is necessary to preserve the correct vertical alignment.

(iii) The second disadvantage of (*a*) is that the placing of the result of multiplication below the multiplicand is a possible source of error; and the second disadvantage of (*b*) similarly is in the correct placing of the later partial products.

§ 10·3. DIVISION.

10·31. *Method I.*

$\cdot00724\div\cdot0892.$

$$\begin{array}{rrl} \cdot0892\,) & \cdot0072400 & (\ 0\cdot081 \\ & 7136 & \\ \hline & 1040 & \\ & 892 & \\ \hline & 148 & \end{array}$$

To find the number of decimal places in the quotient, subtract the number in the divisor from the number used in the dividend.

In such a sum as 375 ÷ ·00075 it is necessary before dividing to make sure by adding zeros to the dividend that there are at least as many decimal places in the dividend as in the divisor.

$$\begin{array}{rrl} \cdot00075\,) & 375\cdot00000 & (\ 500000 \\ & \underline{375} & \end{array}$$

10·321. *Method IIa.* 'Make the divisor a whole number.'

Find ·00724 ÷ ·0892 to three places.

(1) $\frac{\cdot00724}{\cdot0892}=\frac{72\cdot4}{892}$ on multiplying above and below by 10000.

$$\begin{array}{rr} & 0\cdot0811 \\ \hline 892\,) & 72\cdot400 \\ & 71\ 36 \\ \hline & 1\ 040 \\ & 892 \\ \hline & 148 \end{array}$$

The first 0 in the dividend is used to enable the division to be begun so that the 8 has the same place value; this is conveniently shown by writing the quotient above the dividend.

Quotient = $\underline{0\cdot081}$ *Ans.*

(2) $\frac{\cdot00643}{\cdot72}=\frac{\cdot643}{72}.$

$$\begin{array}{r|l} 8 & \cdot643 \\ \hline 9 & \cdot080375 \\ \hline & \cdot00893\ldots\ \textit{Ans.} \end{array}$$

10·322. *Method IIb.* 'Reduce the divisor to Standard Form.'

$$\frac{\cdot00724}{\cdot0892}=\frac{\cdot724}{8\cdot92}\bumpeq\frac{\cdot7}{9\cdot0}\bumpeq0\cdot08.$$

$$\begin{array}{rr} & 0\cdot081 \\ \hline 8\cdot92\,) & \cdot7240 \\ & \cdot7136 \\ \hline & 1040 \\ & 892 \\ \hline & 148 \end{array}$$

The position of the decimal point in the quotient is obtained from the approximate answer, or by dividing ·72 by 8.

10·331. *Method IIIa.* 'Tracking the Unit Digit.'

```
           ·081
        ----------
0·0892 ) 0·007240
             07136
             -----
              1040
             00892
             -----
               148

       23·6
      ------
42·1 ) 993·56
       842
       -----
       151·5
       126·3
       -----
        25·26
        25·26
        -----
```

Here each figure of the quotient has the same place value as that digit of the partial product which comes from the units digit of the divisor, and is placed above it.

10·332. *Method IIIb.*

```
·007240÷·0892

 0·0892
  ·081......
 ----------
 ·007240
   7136
   ----
   1040
    892
    ---

993·56÷23·6.
     23·6
     42·1
    ------
    993·56
    944
    ------
     49·5
     47·2
     ----
      2·36
      2·36
      ----
```

Here the divisor is placed above the dividend (with a space left between them for the quotient) as far to the left as is consistent with it not appearing bigger than the dividend, if the decimal place is ignored.

The first significant figure of the quotient is placed below the units digit of the divisor and the decimal point over that of the dividend.

10·34. *Remarks on the above Methods.*

10·341. *Method I.* (i) The position of the decimal point is obtained as in Method I for multiplication, and there is no preliminary rearrangement of the figures.

(ii) The addition of zeros to the dividend is artificial and its omission is apt to lead to the subtraction being done the wrong way. It is this disadvantage which has caused the method to go out of use.

10·342. *Method IIa.* (i) When, as in Ex. 2, it becomes possible to use short division, this is the simplest method.

(ii) That the first figure of the quotient has the same place value as the last figure of the dividend used at this stage is the same rule as in division of integers, and is already known.

10·343. *Method IIb.* (i) This method specially encourages the use of the rough approximation for it displays a possible single figure divisor.

(ii) As in multiplication, the idea of Standard Form is a useful preliminary to work with logarithms.

10·344. *Method IIIa.* (i) This avoids the preliminary rearrangement of figures, but :

(ii) This is less help than in multiplication, as the rearrangement is needed if a rough approximation is to be used : so that the main advantage gained is an increase in the value of the check.

(iii) The necessity of paying attention to what happens to zeros (marked *0* in the example) which need not be written down at all in either of the other methods is a disadvantage.

(iv) The 8 in the quotient cannot be written down until *after* the partial product *0*7136 has been written down.

10·345. *Method IIIb.* (i) This also avoids the preliminary rearrangement and has the considerable advantage of being exactly similar in form to the corresponding multiplication method (mult. method IVb) ; but

(ii) It involves the difficulty of correct vertical alignment, and suffers also from the disadvantages of Method IIIa, except the last.

10·4. *Method for Rough Estimate.*

$$23\cdot612 \times 42\cdot1 \simeq 20 \times 40$$
$$= 800$$
$$\frac{\cdot00724}{\cdot0892} \simeq \frac{\cdot7}{9}$$
$$\simeq \cdot08.$$

10·5. *Three Notes.*

(I) Multiply many figures by few rather than few by many, e.g. 23·612 by 42·1 rather than 42·1 by 23·612.

(II) Boys who can recognise ·125 as $\frac{1}{8}$ should multiply by $7\frac{1}{8}$, rather than by 7·125 or by $\frac{57}{8}$.

(III) In division, $\frac{46\cdot57}{1\cdot2}$ should become $\frac{465\cdot7}{12}$: slavish use of Standard Form may lead to long division where short division is all that is necessary.

10·6. *Recommendations.*

For Multiplication, Method I should be used.

For Division, Method II*a* or Method II*b* should be used ; preferably Method II*b*, the 'Standard Form' method, except when it is desired to use short division.

One reason why 'Standard Form' has not been recommended for both multiplication and division is that confusion is apt to arise as to the direction in which to move the figures.

10·7. *At a Later Stage.*

When the pupil can rely on the rough approximation to fix the position of the decimal point, the point may be ignored in the working and inserted in the answer.

Those who use the Slide Rule do this.

10·8. *Value of Partial Products. True Remainder.*

It will be noticed that the Committee is recommending a method for multiplication of decimals in which the decimal point is not placed in the partial products, and a method for division in which the decimal point is so placed in the work as not to show the true remainder.

The Committee considers that the boy will by using these methods get the greatest number of these multiplication and division sums right. But he ought to be *able* to find the true partial products or the true remainder, easily at a later stage, or by careful thinking at the 'beginner' stage.

In order to do this for multiplication the idea of the place-value of the digits must be understood, and oral work on the individual items in the product may be useful.

E.g. in $1429{\cdot}64 \times {\cdot}003675$,

the 2×7 is really $20 \times {\cdot}00007 = {\cdot}0014$,
the 1×3 ,, ,, $1000 \times {\cdot}003 = 3$.

In division, to find the true remainder, the boy must learn to understand that, when $\frac{{\cdot}0463}{{\cdot}0071}$ is changed into $\frac{46{\cdot}3}{7{\cdot}1}$, this is done by multiplying above and below by 1000, and that this multiplication affects all the figures that occur in the work, so that the true remainder is obtained by dividing the apparent remainder by 1000. Here also oral work will be valuable.*

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* Reference may be made to the following articles in the *Mathematical Gazette* dealing with multiplication and division of decimals :

F. C. Boon, 'Decimal Processes : "Tracking the Unit",' July, 1930.

R. A. M. Kearney, 'Multiplication and Division of Decimals,' May, 1930—for Methods IV (*b*) and III (*b*) outlined above.

W. F. Sheppard, 'Multiplication and Division of Integers and Decimals,' May, 1931—for methods not mentioned above.

§ 11. DECIMALISATION OF MONEY

11·10. Before any method of decimalising £ s. d., which is applicable only to £ s. d., or which will give only a limited number of decimal places, is taught, boys should have been taught and have mastered the general method which applies to all quantities, and will give as many decimal places as may be required.

£2 3s. 5d. to decimal of £1.

12) 5·00 d.
20) 3·41666... s.
2·17083... £

1 ton 5 cwt. 3 qr. 10 lb. to decimal of 1 ton.

4) 10 lb.
7) 2·5
4) 3·35714286 – qr.
20) 5·83928571 + cwt.
1·29196429 tons.

11·11. They should also know the converse process of multiplying by 20 and 12 for £ s. d., by 20, 4, 28 in avoirdupois weight, and so on.

11·12. When this has been done no other method is really necessary, and it becomes a debatable question whether it is an advantage to teach further a method for decimalising £ s. d. to three decimal places of £1 at sight. Many teachers consider that to do so saves time in the end. But it should not be an added burden taught for its own sake. If with any class it is found to be an added burden it should be dropped.

11·2. *The Three-place Method.*

11·20. The first step is to use £·1=2s., £·05=1s., to obtain the accurate decimal for any whole number of shillings.

These can quickly be learnt by heart through actual repetition.

For the pence, reduce to farthings and use

$$1 \text{ f.} = £·001 + \tfrac{1}{24} \text{ of } £·001.$$

This quickly gives the rule: 'over 3d. add 1, over 9d. add 2.'

Cases in which the shillings and pence do not interfere may be then written straight down, e.g. 17s. 2¼d.=£·859, 14s. 10½d.=£·744. When there is interference, it is best to put down a step at the side:

£3 17s. 8½d.=£3·885 ·85
·035

11·21. For the reading backwards, it will be found necessary to give a little more care to the critical cases 3d. and 9d.

Thus 3d. = £·0125 accurately, and ∴ ≏ £·012 or £·013,
9d. = £·0375 accurately, and ∴ ≏ £·037 or £·038.

$$\begin{array}{llll} & \pounds & & \pounds \\ \therefore\ \text{we have} & \cdot 011 \ = 2\tfrac{3}{4}\text{d.} & & \cdot 036 \ = 8\tfrac{3}{4}\text{d.} \\ & \left.\begin{array}{l}\cdot 012\\ \text{or}\\ \cdot 013\end{array}\right\} = 3\text{d.} & & \left.\begin{array}{l}\cdot 037\\ \text{or}\\ \cdot 038\end{array}\right\} = 9\text{d.} \\ & \cdot 014 \ = 3\tfrac{1}{4}\text{d.} & & \cdot 039 \ = 9\tfrac{1}{4}\text{d.}\end{array}$$

This gives the points where it is necessary to change from 'no subtraction' to 'subtraction' in going back.

Examples. £·819 thought of as £·8 plus £·019 is 16s. 4$\frac{1}{2}$d.
£·993 thought of as £·95 plus £·043 is 19s. 10$\frac{1}{4}$d.

11·22. Some teachers prefer to modify the above method by using the accurate result, 6d. = £·025, and only turning sums less than 6d. into farthings.

11·3. *Use of Three-place Method.*

It is essential that boys should realise that *approximate* decimals must not be used for multiplication without proper safeguards. Their use is for division, and especially for percentages. They may of course be used in such a case as 112/237 of £2 13s. 1$\frac{1}{2}$d., for the net result of the multiplication and division is to reduce the error. It is therefore natural to introduce the 'practice' method for shortening long multiplication, and the three-place decimal method for shortening division and percentages at about the same period of a boy's arithmetical career.

11·4. *Decimalisation at Sight to* 5 *or More Places.*

11·40. For the usual pre-certificate course for ordinary boys this should not be included. If more than three places are needed, the method of division by 12 and 20 should be used.

11·41. At a later stage, however, for those entering a commercial career, the following method will be found to be valuable.

Since a farthing = $1\frac{1}{24}$ mils (a mil = £·001),

$$1\text{d.} = 4\tfrac{1}{6}\text{th mils} : 2\tfrac{3}{4}\text{d.} = 11\frac{2\cdot 75}{6}\text{ mils} = 11\cdot 4583\text{ mils.}$$

$$11\text{s. } 2\tfrac{3}{4}\text{d.} = \cdot 561458\dot{3},$$

$$13\text{s. } 10\tfrac{1}{2}\text{d.} = \pounds\cdot 65 \text{ plus } 42\frac{10\cdot 5}{6}\text{ mils} = \pounds\cdot 65 \text{ plus } \cdot 04375$$
$$= \pounds\cdot 69375.$$

11·42. With a certain amount of practice the operations indicated above can be performed mentally, and when this stage has been reached the method has become one of much value. It is not of value unless it is used so frequently as to become automatic.

§ 12. THE FOUR RULES WITH OTHER COMPOUND QUANTITIES

12·0. The methods applicable to the four rules in money are equally applicable here, but the actual computation is usually more involved. Some remarks are appended for dealing with such a sum as 15 tons 14 cwt. 3 qrs. of goods at £5 16s. 4d. per ton, a type very common in the commercial world.

12·1. *Method* 1.

Usual practice method, working correct to 3 places of decimals in the pence.

	£	s.	d.
1 ton	5	16	4
3 tons	17	9	0
15 tons	87	5	0
10 cwt.	2	18	2
4 cwt.	1	3	3·2
2 qrs.		2	10·9
1 qr.		1	5·45
15 tons 14 cwt. 3 qrs.	£91	10	9·55
	£91 10s. 9½d.		

Vulgar fractions should not be used in the pence column.

12·2. *Method* 2.

In this method the cost of 15 tons 14 cwt. 3 qrs. is found at £1, at 1s., and at 1d. per ton, and then at amount stated as shown.

Cost of 15 tons 14 cwt. 3 qrs. :

at £1=£15 14s. 9d. ; at £5 =£78 13 9
at 1s.= 15s. 8·85d. ; at 16s.= 12 11 9·6
at 1d.= 1s. 3·7375d. ; at 4d. = 5 2·95
£91 10 9·55

This method can become awkward for young boys since it requires a knowledge of the tables up to 19 times or else means somewhat heavy multiplication.

12·3. *Method* 3.

Practice—decimal method, working to five places of £.

£5 16s. 4d. = £5·816667 –

	£
1 ton	5·816667 –
10 tons	58·16667 –
5 tons	29·08333
15 tons	87·25000
10 cwt.	2·90833
4 cwt.	1·16333 +
2 qrs.	·14542
1 qr.	·07271
15 tons 14 cwt. 3 qrs.	£91·53979, i.e. £91·540 = £91 10s. 9½d.

The Committee recommend Method 1 as that most often used. It should be learnt first ; Method 3 should be learnt later.

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e.g. ·1, ·08, using the reverse argument that the number of decimal places in the square will be twice the number in the root.

13·40. Further explanation of the general method may be deferred; the remaining work should be done at first in a purely mechanical fashion and the result justified by subsequent squaring.

The first examples will, of course, 'come out,' and the next stage will be to give exercises which do not. It is suggested that an arrangement of the work at first may be :

13·41. *To find the square root of* 444·113476.

	2	1	·0	7	4	
	4	44	·11	34	76	
2	4					
41		44				..............(i)
		41				
4207		3	11	34		..............(ii)
		2	94	49		
42144			16	85	76	
			16	85	76	

13·42. Reference to a figure, or to algebra, should be made later to explain the reasoning underlying the process. A boy should know that there *is* a simple explanation beneath an apparently complex 'working.'

It is worth while to point out that, for example, at line (i) in the worked example, the 44·113476 is 444·113476—20^2, that at line (ii) the 3·113476 is 444·113476—21^2, and so on.

13·5. Special attention should be given to finding square roots of decimals with one significant figure, e.g. $\sqrt{\cdot 1}$, $\sqrt{\cdot 6}$, $\sqrt{\cdot 06}$, $\sqrt{\cdot 0005}$. Applications of Pythagoras' Theorem to plane and solid figures make useful exercises.

13·6. It may come as a surprise to a boy to find that a line $\sqrt{2}$ inches in length can be drawn on paper but cannot conceivably be measured with a ruler.

It is useful to emphasise that to say

'$\sqrt{2}=1{\cdot}414\ldots$' is a rough way of saying that

'$\sqrt{2}$ lies between 1·414 and 1·415.'

13·7. In dealing with the square roots of fractions, such as $\frac{1}{2}$, $\frac{1}{3}$, etc., it is better to convert to a fraction whose denominator is a perfect square, e.g. $\frac{1}{2}=\frac{2}{4}$ or $\cdot\frac{5}{1}$, whichever is convenient, and therefore

$$\sqrt{\frac{1}{2}}=\frac{\sqrt{2}}{2} \text{ or } \frac{\sqrt{\cdot 5}}{1}.$$

§ 13. 'SQUARE ROOT'

13·1. Before any work is attempted on square root the idea of a square must be thoroughly understood. Thus $7 \times 7 = 49$, or the square of $7 = 49$, or $7^2 = 49$, or again $7 = \sqrt{49}$. The squares of the first 20 natural numbers should be learnt by heart. In teaching pupils how to obtain the square root of any number, the method of finding factors which are squares comes first. Thus, if 2 is a factor of the original number, try 4; if 3 is a factor, try 9, and so on.

Thus

$$1764 = 4 \times 441 = 2^2 \times 3^2 \times 7^2 :$$

$$\therefore \text{ square root of } 1764 = 2 \times 3 \times 7 = 42.$$

The index notation may be used: boys appreciate its convenience and use it without difficulty.

13·2. In finding the square root of a fraction, the point to stress in class teaching is that a mixed number should be reduced to an improper fraction.

The discussion of a simple example, e.g.

$$\sqrt{1\tfrac{9}{16}} \text{ which is } \sqrt{\tfrac{25}{16}} = \tfrac{5}{4} = 1\tfrac{1}{4},$$

will avoid such mistakes as writing

$$\sqrt{1\tfrac{9}{16}} = 1\tfrac{3}{4} \text{ or } \sqrt{1\tfrac{25}{144}} = 1\tfrac{5}{12}.$$

The test of squaring $1\frac{3}{4}$ and comparing with $1\frac{9}{16}$ is useful.

This device of squaring also helps later in making clear such relations as $\sqrt{\frac{3}{4}} = \dfrac{\sqrt{3}}{2}$, $\sqrt{15} = \sqrt{3} \times \sqrt{5}$.

13·3. In teaching the general method special attention should be given to the rule for marking off the digits in pairs. Boys should have their attention drawn to the fact that if a number has 1 or 2 digits its square root has one digit; if it has 3 or 4 digits its square root has 2 digits, and so on.

In general, after marking off pairs of digits from the decimal point, there will be as many digits in each part (whole number and decimal) of the square root as there are pairs in the corresponding number.

e.g. 4/43/·94/49 square root is 21·07,

4/·43/94/49 square root is 2·107.

This will explain the first step in the general method. It will enable certain square roots, e.g. $\sqrt{\cdot 01}$, $\sqrt{\cdot 0064}$, $\sqrt{1 \cdot 44}$, to be written down, but is more useful as a device for squaring mentally,

§ 14. LOGARITHMS

14·0. The use of tables is attractive to most boys, and even the more stupid among them can be taught without difficulty the use of logarithms for doing straightforward sums. It is all the more important that the subject should be introduced in such a way that as many as possible should understand the principle on which they are based, and all should be trained in neat arrangement and the avoidance of glaring misstatements.

14·11. There are two main ways in which the logarithm table may be introduced. In the first the table is regarded as a conundrum to be solved. The comparison of results of very simple operations with entries in the table provides clues from which the *modus operandi* is apparent; and their whole use for the purposes of computation can be developed from the simple discovery that $\log (ab) = \log a + \log b$, with its corollaries that $\log (a/b) = \log a - \log b$, and $\log (a^n) = n \log a$. This method makes no attempt to indicate the representation of the number as a power of 10, only bringing it out incidentally from the observed facts that

$$\log 10 = 1, \text{ and that } \log 8 + \log 2 = 1 + \log 1{\cdot}6.$$

14·12. For this reason others prefer to develop the subject in a way that introduces the index law from the outset. The graph of 10^x is plotted for the values $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, and thence for $x = \frac{3}{4}, \frac{7}{8}$, etc., by using the index laws $\sqrt{10^a} = 10^{a/2}$, $10^a \times 10^b = 10^{a+b}$, derived as a generalisation from the case where a and b are integers. With this graph, numbers can be expressed as powers of ten, and their products and quotients obtained to a two-figure degree of accuracy; subsequently, recourse is had to the logarithm table as a fuller and more accurate source of the same information. Those who follow this method will find some preliminary practice in simple work with indices essential. The two methods are not mutually exclusive; if the first has been used for a preliminary introduction, the second will add interest to a later revision course. It should not be used as the only method unless logarithms are introduced at a fairly late stage.

14·2. Some time should be devoted to the mere use of tables, and to the logarithms of numbers greater than 10. Boys who have been trained in Standard Form will find the latter easy to understand; but to none do they present serious difficulty.

14·3. A method of arrangement of work should be inculcated at this stage, before numbers less than 1 are tackled. The following is suggested, it being understood that the actual writing of numbers as powers of 10 may have been done at the early stages but found too clumsy for permanent use :

$$\frac{(21{\cdot}2)^3 \times 3{\cdot}234}{51{\cdot}7 \times \sqrt{2{\cdot}8}} = 356$$

Number	Logarithm	
$(21{\cdot}2)^3$	$1{\cdot}3263 \times 3 = 3{\cdot}9789$	
3·234	0·5097	
Num.	4·4886	4·4886
51·7	1·7135	
$\sqrt{2{\cdot}8}$	$0{\cdot}4472 \times \frac{1}{2} = 0{\cdot}2236$	
Denom.	1·9371	1·9371
356		2·5515

The work should be planned systematically from the beginning and too great compression discouraged.

14·4. For the logarithms of numbers less than 1, the advantage of the conventional notation is easily made apparent to all boys : it is better that they should speak at first of $\bar{2}{\cdot}3$ as 'negative two point three,' or as 'minus two plus point three,' than as 'bar two point three,' and that they should be trained in such exercises as

$$\bar{2}{\cdot}326 = -1{\cdot}674,$$

and the converse step. Ample practice should be given in such sums as $\bar{2}{\cdot}4 + 3{\cdot}2$, $\bar{2}{\cdot}9 + \bar{2}{\cdot}2$, $\bar{2}{\cdot}4 - \bar{1}{\cdot}2$, $\bar{2}{\cdot}4 - \bar{1}{\cdot}6$, $\bar{2}{\cdot}4 \times 4$, $\bar{2}{\cdot}4 \div 4$, each of which presents a new difficulty. In a class of boys, some of whom subtract by equal additions and some by decomposition, care is needed to show that either method is applicable to this type of number.

14·5. The work involved in such sums as $(0{\cdot}3)^{0{\cdot}3}$ and $(0{\cdot}3)^{-0{\cdot}3}$ needs careful explanation if taken at this stage and demands a very clear understanding of the notation. The arrangement of the work will not differ from the example above.

$(0{\cdot}3)^{0{\cdot}3}$	Number	Logarithm
$= \underline{{\cdot}697.}$	$(0{\cdot}3)^{0{\cdot}3}$	$\bar{1}{\cdot}4771 \times 0{\cdot}3 = \begin{cases} +0{\cdot}14313 \\ -0{\cdot}3 \end{cases}$
	·697	$\bar{1}{\cdot}8431$
	$(0{\cdot}3)^{-0{\cdot}3}$	$\bar{1}{\cdot}4771 \times -0{\cdot}3$ $= 0{\cdot}5229 \times 0{\cdot}3 = 0{\cdot}1569$ or $\bar{1}{\cdot}4771 \times -0{\cdot}3$
$(0{\cdot}3)^{-0{\cdot}3}$		$= \begin{cases} +0{\cdot}3 \\ -0{\cdot}14313 \end{cases}$
$= \underline{1{\cdot}435.}$	1·435	·1569

14·6. At some stages a boy may find interest and scope for ingenuity in constructing portions of the tables for himself, e.g.

$$2^{10} = 1024 \simeq 10^3,$$

$$2 \simeq \sqrt[10]{10^3} \simeq 10^{0\cdot3}.$$

Log 2 having been found, the logarithms of 3, 7 and 11 can be found by using

$$3 \simeq \sqrt[4]{80}, \quad 7 \simeq \sqrt[4]{2400}, \quad 11 \simeq \sqrt{120}.$$

The logarithm of any number to base 10 can be calculated in this primitive fashion, and this was in fact one of the methods originally used by Briggs. Probably it is the only method for calculating logarithms to base 10 directly and without involving explicitly or implicitly natural logarithms.*

If log 2, log 3, log 7, and log 11 be accurately supplied, a very considerable number of logarithms may be found, such as the logs of 96, 98, 99, 100, 105, 108, 110, 112, (to take a small section of the integers).

14·7. *Notes.*

1. It is very much easier to add or subtract 'up and down' than 'across.'

2. More mistakes are made by copying wrongly from the tables (e.g. 4722 in mistake for 4772 or 2837 for 2387), than in failure to use the right method.

3. Log 2188 is best read by subtracting ·0004 from log 2190, and not by adding ·0016 to log 2180.

4. Antilogarithm tables are often omitted to avoid the danger of confusion with the logarithm tables and on the ground that they suggest a fourth-figure accuracy which they cannot always give; the 'backward' use of logarithm tables needs practice.

5. A boy should never have to keep one finger in the tables and one in the body of the book: separate books of tables are of great advantage.

6. Opportunity should be given to boys at least to see, if not to use, 3-figure, 5-figure or 7-figure tables, and learn that there is no magic in the 4-figures normally supplied.

7. Questions of the type $(\log 2)^3$, or $10\sqrt{\log 3}$ are better postponed to a much later stage. The mechanical and intelligent use of logarithms is quite easy and very valuable, but an ability to evaluate even $10^{\log 5}$ quickly and accurately is beyond the ordinary boy.

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* A fuller description of the method will be found in articles in the *Mathematical Gazette* by Dr. Henderson (Dec. 1930) and W. C. Fletcher (Jan. 1931).

8. Practice should be given in substitution in formulae ; especially valuable are the type in which multiplication and addition are mixed, e.g. $\frac{a^2+b^2-c^2}{2ab}$.

9. Interest in the 'bar' notation may be fostered by using it in other and less familiar ways, e.g.

(1) $997+999=100\bar{3}+100\bar{1}=200\bar{4}=1996$

(2) $999\times 997=996003$

$$\begin{array}{r} 100\bar{1} \\ 100\bar{3} \\ \hline 100\bar{1} \\ \bar{3}003 \\ \hline 100\bar{4}003 \\ \hline \end{array}$$

10. The most difficult thing to learn in the use of four-figure tables is how to decide when the fourth figure of the answer is trustworthy. Beginners should always be relieved of this decision by being told whether to give up an answer correct to three figures or four.

Unless the full number of figures that can be obtained from the tables in use are required at first, and occasionally afterwards, boys will only too readily suppose that the fourth figure 'doesn't matter' ; their sense of responsibility for the accuracy of their work will be damaged—a serious undoing of one main purpose of teaching arithmetic.

11. With more experience, boys should learn the general principles on which we decide whether to trust the fourth figure or not. Two relevant questions are : (*A*) Do we really know the four-figure logarithm ? (*B*) If we do, does that fix the four-figured number ? The answers to these questions are too long to give fully here. As a first approximation to an answer we suggest : (*A*) Probably, if we have added or subtracted six logarithms or less which are different from each other ; (*B*) Probably, if the first two figures of the answer are less than 44 ; after this point in the table we have two four-figured numbers sharing one four-figured logarithm between them.

The following are examples of the unreliability of figures obtained from tables :

$$\log 9{\cdot}319=\cdot 9693,\quad \text{also } \log 9{\cdot}317=\cdot 9693.$$

This would have been all right if 9319 had been obtained from 9320, using a negative difference.

$$\begin{aligned} \text{Again } \log 9{\cdot}321&=\cdot 9694=\log 9{\cdot}319, \\ \text{and } \log 9{\cdot}315&=\cdot 9691=\log 9{\cdot}313, \\ \log 9{\cdot}255&=\cdot 9663=\log 9{\cdot}253. \end{aligned}$$

§ 15. UNITARY METHOD, RATIO AND PROPORTION

15·1. Two quantities of the same kind may be compared in respect of size in two ways: Near the end of a race a competitor is concerned with the distance which separates him from other competitors, i.e. with differences; on the other hand, it would serve little purpose to say that the model of a ship is 500 feet shorter than the ship; what is useful is the statement that the length of the model is 3/100 (or on the scale of 3 ft. to 100 ft.) of the ship. Here the comparison is made by use of a ratio. The latter method is by far the more general. Even young children are more prone to think in terms of 'so many times as big as' than in terms of 'so much bigger than.'

15·2. Problems in proportion are the commonest in arithmetic, and practically all applications of arithmetic to shop sums, percentages, to quantitative work in chemistry and elementary physics are questions of proportion, i.e. of direct, or more rarely, inverse variation. It is of vital importance that the pupil should be able to approach these questions with confidence and solve them with accuracy.

15·30. Two methods, the *unitary* method and the method of *ratio*, are used. Both are strongly supported. They are illustrated by the following examples:

15·31. *Unitary Method.*

Example 1. If four bats cost £10, how much will seven bats cost?

$$\begin{aligned}&\text{Cost of 4 bats is £10,}\\&\text{,, ,, 1 bat ,, } \frac{£10}{4},\\&\text{,, ,, 7 bats ,, } \frac{£10}{4}\times 7.\end{aligned}$$

Example 2. A certain field can be mown by 7 men in 2 days, how long will it take 4 men?

$$\begin{aligned}&\text{7 men can finish it in } 2 \text{ days,}\\&\text{1 man ,, ,, ,, } 2\times 7 \text{ days,}\\&\text{4 men ,, ,, ,, } \frac{2\times 7}{4} \text{ ,,}\end{aligned}$$

It is clear that 1 man will take longer than 7 men, and that we therefore multiply in line 2.

Example 3. If $2\frac{1}{4}$ yds. of cloth cost 17s. 6d., what do $3\frac{1}{2}$ yds. cost ?

Cost of $2\frac{1}{4}$ yds. is 17s. 6d.
$\therefore$,, ,, 1 yd. ,, $\dfrac{\text{17s. 6d.}}{2\frac{1}{4}}$;
$\therefore$,, ,, $3\frac{1}{2}$ yds. ,, $\dfrac{\text{17s. 6d.}}{2\frac{1}{4}} \times 3\frac{1}{2}$.

Example 4. If a train goes p miles in q hours, how far will it go in r hours ?

In q hours it goes p miles,
,, 1 ,, ,, ,, $\dfrac{p}{q}$,,
,, r ,, ,, ,, $\dfrac{p}{q} \times r$ miles.

15·4. *The Method of Ratio.**

Example 1. 'If 4 bats cost £10, what will 7 cost ? ' is worked as follows :

$\because$ the number of bats is increased in the ratio $\frac{7}{4}$,
$\therefore$ the cost is increased ,, ,, $\frac{7}{4}$,
and is £10 × $\frac{7}{4}$.

Example 2. 'A certain field is mown by 7 men in 2 days, how long will it take 4 men ? '

$\because$ the number of men is decreased in the ratio $\frac{4}{7}$,
$\therefore$ the time required is increased ,, ,, $\frac{7}{4}$,
and is 2 days × $\frac{7}{4}$.

15·41. By drill of this kind the pupil becomes familiar with the use of a ratio as a multiplier and learns to distinguish between (1) *direct variation*, the test for which is that the dependent variable is doubled (or halved) if the prime variable is doubled (or halved), and (2) *inverse variation*, the test for which is that the dependent variable is halved (or doubled) if the prime variable is doubled (or halved).

15·42. Whenever he is in difficulty he should apply the test. He comes to a stage when he need not write down the argument, but he can multiply straight off, by the appropriate ratio.

Thus *Example* 3. 'If $2\frac{1}{4}$ yds. of cloth cost 17s. 6d., what do $3\frac{1}{2}$ yds. cost ? '

The cost is 17s. 6d. $\times \dfrac{3\frac{1}{2}}{2\frac{1}{4}}$

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* Remember that × is to be read 'multiplied by' not 'times.'

and *Example* 4. 'If a train goes p miles in q hours, how far will it go in r hours?'

$$\text{It will go } p \text{ miles} \times \frac{r}{q}.$$

15·43. The method may be introduced by recalling results of such examples as 1 and 2, done by unitary method, and showing that $\frac{£10}{4} \times 7$ and $\frac{2 \times 7}{4}$ days may be written $£10 \times \frac{7}{4}$, and 2 days $\times \frac{7}{4}$. But it is preferable not to appeal to the mechanism of the unitary method as a help, but to rely on the intuition of proportionality, which practice in the unitary method has helped to create.

15·5. The work should be accompanied by graphical illustration, as e.g. a money-quantity graph for a constant buying rate, for direct variation; and a time-speed graph for a constant distance, for inverse variation. It should be correlated with some scale drawing, and it may be correlated with a simple treatment of similar triangles whereby it will lead experimentally to the laws of (i) direct squares, and (ii) direct cubes for the (i) areas, and (ii) volumes of models.

15·6. The use, or rather the abuse, of what is called the *Rule of Three*, in which the pupil is expected to solve the first of these examples by writing down mechanically, 'as 4 : 7 :: £10 : the answer,' is strongly deprecated.

15·7. A tabulation of the data, especially in the cases of Compound Proportion, enlists the eye in the service of the mind.

Example 5. What principal invested at $2\frac{1}{2}$ % Simple Interest will earn £24 in three years?

Tabulation.

	Principal	Time	Interest
Standard Case	- £100	1 yr.	£$2\frac{1}{2}$.
Actual Case -	- £x	3 yrs.	£24.

15·71. Solution using unitary method:

£$2\frac{1}{2}$ is the interest for 1 yr. on £100,
£1 ,, ,, ,, 1 ,, ,, $\frac{£100}{2\frac{1}{2}}$, i.e. £40,
£24 ,, ,, ,, 1 ,, ,, £40×24,
£24 ,, ,, ,, 3 yrs. ,, $\frac{£40 \times 24}{3}$, i.e. £320.

15·72. Solution using the ratio method.

The interest is increased in the ratio $\frac{24}{2\frac{1}{2}}$, i.e. $\frac{48}{5}$;

$\therefore$ the principal is increased in the ratio $\frac{48}{5}$.

The time is increased in the ratio $\frac{3}{1}$;

$\therefore$ the principal is decreased in the ratio $\frac{1}{3}$.

The principal is £$100 \times \frac{48}{5} \times \frac{1}{3}$, i.e. £320.

The argument would eventually be mental; only the last line would then be written down.

15·73. *Example* 6. How much must a man invest in 5 % Stock at 103½ to produce a net annual income of £400 after paying Income Tax at 4s. 6d. in £1 ?

Tabulation.

	Cash Paid	Stock Bought	Gross Income	Net Income
Standard Case -	£103½	£100	£5	£$5 \times \frac{31}{40}$
Actual Case -	£x	—	—	£400

The tabulation shows to the eye the quantities, viz. those in the first and last columns, with which the question asked is concerned. They are in proportion.

§ 15·8. COMPARISON OF THE TWO METHODS.

15·81. *The Unitary Method* or the *Method of Reduction* to *Unity* is probably most generally used. It is clearly the better method for the beginner.

15·811. The name is natural as defining its method, but does not adequately express its essence. The German *schlussrechnung* is more expressive—calculation by inference or deduction—for the essence of the method is that the argument is expressed step by step and at length.

15·812. Stating the argument involves the necessity of conscious thinking and the process of stating the facts clearly and in their proper order (i.e. with the quantity-to-be-found at the end) is a good exercise in simple composition, and on no account should a slovenly statement be tolerated.

15·813. The great merit of the method is that the simplicity of the thinking involved makes it suitable for young and slow pupils. Their experience is in connection with transactions and quotations in which a unit is prominent: thus, they buy apples at so much a pound, sweets at so much a quarter (lb. understood); their parents buy petrol at so much a gallon; they see speed-limit notices of so many miles per hour. To reduce to unity is not only to put them on to familiar ground, but also to split the sum into two simpler sums, in one of which there is a simple operation of multiplication, in the other a simple division.

15·814. In the early drill the examples should require only operations with integers. If they are simple and fit one another nicely, as in Examples 1 and 2, perception is so immediate that the process is readily grasped. When the nature of the process no longer gives him trouble, the pupil proceeds to harder examples and can concentrate his attention on the difficulties inherent in using fractional and algebraical quantities.

15·815. The method has these disadvantages: (1) In certain cases of compound proportion it becomes cumbrous and tedious. (2) It keeps the fundamental notion of ratio in the background. These disadvantages make teachers anxious to introduce the ratio method as soon as the pupil is mature enough to receive it. An able boy passes of himself from unitary method to working directly by ratios. For most, the learning, or the transition to the ratio method, will require careful and methodical teaching. It should not be attempted without a full realisation of its difficulties.

15·82. The *Ratio Method*, in common with the unitary method, has the advantage that in the initial stages of instruction it demands conscious thought by requiring the pupil to write his argument. By dispensing with the unit step it effects an economy, especially when used in cases of compound proportion. There is subsequently the greater economy in that, when the pupil is at home with the method, he can write down the solution in one line by operating with a set of multipliers. Finally, its conspicuous advantage is that it forms a habit of thinking in terms of ratio and of using ratios as multipliers, which simplifies his work in arithmetic and the introductory work in trigonometry and mechanics, and which enlarges his mathematical outlook generally.

15·821. Practically all children have some idea of relative magnitude. They accumulate a store of experience of proportionality from such commercial transactions as they come into contact with: from map-reading and drawing, from model-making (e.g. Meccano), and even from such things as a child's attempt to draw a house, or a boy's fondness for drawing motor bicycles. This experience contributes to building up an intuition which is neither generalised nor definitely formulated. To reduce this experience to formal statement is an obvious piece of education and is of the highest value.

15·822. The pupil has no difficulty as long as the ratio involved is of the type 3 inches to 1 mile, or is of the form $m:1$ or $1:m$, where m is an integer. His difficulty is to see that the same argument holds and the same operation must be performed in the case of ratios of the form $m:n$. The same difficulty has already been encountered in fractions, where $\frac{3}{5}$ of $\frac{17}{6}$ has presented a difficulty to the pupil who had little with 3 times $\frac{17}{6}$ or with $\frac{1}{5}$ of $\frac{17}{6}$. In the case of ratios, the difficulty is overcome by the kind of drill indicated above.

15·9. *Notes.*

(1) It is not without significance that every measured quantity implies a ratio. Every measured quantity is expressed as so many times (not necessarily an integral number of times) a standard quantity. So that while the conscious use of the notion is introduced somewhat late in the arithmetical course, the notion itself has been involved in the earliest arithmetical topics.

(2) When considering the advisability of introducing the ratio method, teachers will be well advised to find out what method is in use for proportion questions in the science class-room, and they will aim at coordination with the science master in their treatment of the subject.

(3) The fact that most arithmetical problems are cases of direct or inverse variation is apt to make the pupil apply the method of proportion in cases where it does not apply. It is worth while, at an early stage, to consider questions in which there is either (I) no law or (II) a different law of variation, e.g. (I) if a boy of 10 eats three meals a day, how many meals does a man of 40 eat? (II) If a rubber ball of three inches diameter weighs 4½ ozs., what would be the weight of a ball of the same composition of two inches diameter?

(4) As in the case of other doublets (e.g. palsy and paralysis), the words 'rate' and 'ratio' have allied meanings, but the modified form 'rate,' being current in popular speech, is connected with commoner notions, while the directly borrowed word 'ratio' has a more technical application (the French word *raison*, derived from the same Latin word 'ratio' is used with both meanings). For division of like quantities, 'ratio' is used; the fraction $\frac{9}{2}$ expresses the ratio of 9 hours to 2 hours or 9 miles to 2 miles. For division of unlike, 'rate' is used: the rate of $\frac{9}{2}$ miles per hour expresses the distribution of 9 miles over 2 hours.

(5) Cases of simple proportion may be regarded in two ways: (I) as cases of equality of rates, or (II) cases of equality of ratios. Thus, if 4 bats cost £10 and 7 bats cost £17 10s., we have a case of proportion because:

(I) In each case the rate of buying is £2 10s. per bat.

(II) The ratio of the purchased quantities equals $\frac{7}{4}$ which equals the ratio of the payments made.

(6) The term 'ratio' does not admit of a mathematical definition and the teacher had much better not try to give one. It is sufficient for elementary purposes that a fraction can be used to express its value.

(7) The first introduction of the word 'ratio' should perhaps be in some problem where more than two magnitudes are involved, e.g. the division of a legacy into parts in the ratios 4 : 5 : 6, for here the advantage of using the new word is apparent. The teacher can

make a habit of using it along with a synonymous phrase until the pupil is familiar with its real meaning, e.g. 'A quantity is doubled, i.e. increased in the ratio 2 : 1,' and so on.

It will thus become clear, that 'to increase X in the ratio of 8 : 5' means that X must be multiplied by $\frac{8}{5}$. The inexact phrase 'increase in the proportion of 8 : 5' should never be used, or if used, it should be corrected.

(8) If we say that two quantities are proportional, we imply that the magnitude of each is susceptible of variation, and we mean that the ratio of any two values of the one is equal to the ratio of the corresponding values of the other. Thus we say that the circumferences of circles are proportional to their diameters, because the ratio of the circumferences of any two circles is equal to the ratio of their diameters.

It is a misuse of the word 'proportion' to say '6d. : 10d. is in the proportion of 3 : 5.' This popular use of the word should not be permitted ; the right word is 'ratio.'

(9) Where the ratio method is used, the teacher should frequently use the terms 'direct' and 'inverse' ratio, and so habituate the class to the right use of them.

(10) In using the unitary method, it should be pointed out that the 'unit' may be a larger unit, as a score, or even a fractional unit, as half-a-pound.

Thus, 'if 21 lb. cost 2s., how much will 56 lb. cost ?' Here the unit could be conveniently taken as 7 lb.

(11) Cases of compound and of inverse proportion can often be simplified by the use of a compound unit. Thus, the 'ton-mile' is used habitually in practice ; a 'ration' is the amount of food allotted to a soldier for a day ; a 'man-hour' can be used for the amount of work done by one man in one hour.

'If 15 men do a job in 8 hours, how long should it take 20 men ?' The job consists of 8 times 15 men-hours, each of the 20 men must do $\frac{15 \textit{ times } 8}{20}$ of these units, i.e. 6 hours are required.

(12) The equivalence of the phrases (i) 2s. 6d. in the £1 on ... , (ii) $1\frac{1}{2}$d. in the 1s. on ... , (iii) $\frac{1}{8}$ of ... , (iv) 0·125 of ... , (v) $12\frac{1}{2}$ % of ... , (vi) in the ratio of 1 : 8 to ... should be recognised.

(13) Attention should be given to preventing such statements as :

£5 10s. pays the fares of 22 people,

£1 ,, ,, ,, $\frac{22}{\text{£5 10s.}}$ people,

or even $\frac{22}{\text{£}5\frac{1}{2}}$

§ 16. MENSURATION

16·0. Mensuration has been described as one half of arithmetic. It is the application of arithmetic to geometrical aspects of concrete things which are easily open to observation, if not already familiar. Its successful pursuit involves the development of geometrical perception by observation and drawing.

16·10. It is essential that clear conceptions of the units used should be in a boy's mind whether he is dealing with English or metric measures. Most teachers quote the length of a cricket pitch as illustrating a chain, but few realise that a full-sized football ground (120 yds. × 80 yds.) is rather less than 2 acres. Eye-estimation should be encouraged.

16·11. It is important to stress those units which are in use, e.g. the land surveyor's rod, the litre, omitting those rarely required.

16·12. Some account of the history of the commoner English measures, e.g. the furlong, the hundredweight, is usually of interest to boys.

16·13. Simple connections amongst units, e.g. that 1 chain is $\frac{1}{10}$th of a furlong and is the length of a cricket pitch and that 1 acre is 10 sq. chains, should be noted. Also relations between volume and capacity. Few boys know that 1 gallon is legally defined as the volume of 10 lb. of water, or that 1 cu. ft. of water weighs about 1000 ozs. Simple connections between English and metric units, e.g. pints and litres, should be obtained.

16·14. Omission of units in any part of written work should not be tolerated.

16·2. Pythagoras' Theorem, approached through the '3, 4, 5' triangle and a simple dissection of squares, may be introduced early to add variety to the examples. Particularly valuable are three-dimensional exercises requiring this theorem.

16·3. From the area of the rectangle are derived successively rules for the area of the parallelogram, triangle, and trapezium. It should be emphasised that any side of a parallelogram or triangle may be taken as 'base.' Particular attention should be paid to those figures in which an altitude falls outside the base. To calculate the perpendicular on to the hypotenuse of a right-angled triangle having two sides known, is a good exercise on finding an area in two ways. Various methods of finding an area of a trapezium should be shown, but *one* learnt. A boy should be able to 'prove the rule.' He should see that writing the formula as $\frac{h(a+b)}{2}$ shortens arithmetical computation.

16·40. The connection between the circumference and diameter of a circle may be regarded as an illustration of proportion. When it has been suggested that these are proportional, various methods may be used to find the constant.

16·41. It should be made plain that π does not mean $\frac{22}{7}$, but that π may be replaced by different numbers according to the degree of accuracy in the data. Boys are always interested in the history of π.

16·42. The more elementary exercises on the circle and cylinder can all be taken from everyday life—e.g. the length of a roll of paper of thickness 0·1 mm., given the diameter of the roll, the time taken to roll the cricket field.

16·43. This work is a good starting point for such algebraical ideas as dimensions—e.g. by comparing the formulae $C=2\pi r$, $A=\pi r^2$, $S=2\pi rh$, $V=\pi r^2h$—the advantages of factorising—e.g. in $T=2\pi rh+2\pi r^2$, $A=\pi a^2-\pi b^2$; and simple transformation of formulae.

16·44. Boys need training not to put in the numerical value for π until absolutely necessary.

16·50. The volume of a pyramid may be approached by considering the volume of each of the six pyramids obtained from a cube, of edge a, by drawing its diagonals.

16·51. Each pyramid is of volume $\frac{1}{6}a^3$, which may be written $\frac{1}{3}a^2 \cdot \frac{1}{2}a$, and by regarding a pyramid as composed of thin slices all parallel to the base, the usual argument ensues that pyramids of the same vertical height and on equal bases have equal volumes, leading to the general formula $V=\frac{1}{3}bh$.

16·52. An excellent but difficult exercise in visualisation is the diagram used to show the right prism and a pyramid of equal height on the same triangular base. Here, as elsewhere, the construction of models is helpful. The volumes of the cone and the sphere will be deduced from this when the area of the surface of a sphere has been found.

16·61. To obtain the area of the surface of a sphere, boys may be told the fundamental theorem connecting a zone of a sphere with the corresponding zone of the circumscribed cylinder. Exercises on the Earth assist geographical teaching.

16·62. The surface of the cone may be obtained by developing into the area of a sector of a circular piece of paper.

16·63. Exercises such as 'Find the radius of the sphere whose volume is 1 cubic foot' give useful logarithmic work involving the calculation of the cube root of a decimal.

16·71. Every opportunity of linking up areas or volumes of the simpler solids should be made the most of: e.g. 'In the figure, prove the volumes of cylinder, sphere and cone are as 3 : 2 : 1, and find the corresponding ratios for the areas.'

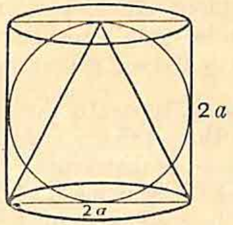


Fig. 1.

16·72. If time permits, valuable work involving every branch of elementary mathematics, including trigonometry, may be done on sectors and segments of circles and spheres, circular rings and zones, shells of spheres and frusta of cones.

16·8. The mensuration of the simpler regular solids may also be taken; and as irregular areas occur very often in practice, some mention of the mid-ordinate rule, or Simpson's rule, seems desirable.

16·9. Every opportunity should be taken to remember formulae and to check formulae by considering dimensions; also to emphasise that areas are proportional to the squares, and volumes to the cubes, of linear dimensions.

§ 17. GRAPHS

17·0. Graphs will enter into every branch of school mathematics and into correlated subjects such as physics and geography. They should not be regarded as a subject to be taught exhaustively at a definite stage. Graphical work may well be done on loose sheets that can be fixed in any exercise book required.

17·1. The aim throughout should be to illustrate vividly the dependence of one variable upon another, looking forward to the calculus rather than to analytical geometry.

17·2. It is important to separate the difficulties arising in the reading of graphs from those arising in drawing graphs. Boys can use graphs shown to them, or distributed to them, long before they acquire the skill needed for drawing a graph. This skill is stimulated by exhibiting really good work on a notice board in the classroom.

17·31. The earliest work may well be the illustration of changes in quantities automatically recorded or easily measured, the self-recording barometer for instance, or perhaps shadow lengths or plant heights. For this, ordinary plain paper is adequate, and indeed preferable.

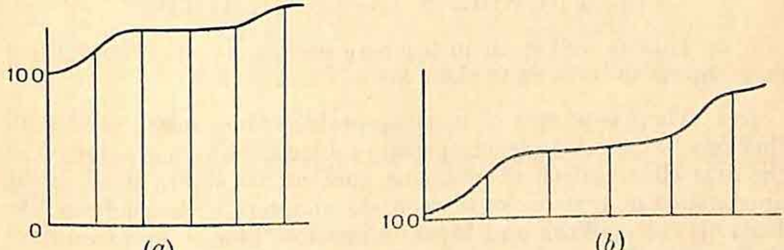
17·32. When the changes are not automatically recorded from moment to moment we plot a series of uprights and the question soon arises: Are we able to obtain information by drawing intermediate uprights? The distinction between column or comparison graphs, e.g. those illustrating annual rainfall, and graphs of continuously changing quantities should be made.

17·33. For measuring intermediate uprights, it will soon be seen that *one* graduated upright is enough, but that squared paper is a great help.

17·41. If the scale divisions are clearly marked at convenient regular intervals along the axes, there is no need for the scales to be stated elsewhere, and no additional credit should be given if they are. But if scales *are* stated elsewhere, badly expressed statements, such as 1 sq. cm. = 1000 gms., cannot be too severely condemned.

17·42. It should be brought out by suitable examples that it is not necessary to measure from zero on either axis.

17·43. Later, boys should realise that omission of scale divisions on the axes of a graph may be seriously misleading. Changes may be exaggerated by removing a portion of the ordinates so that such a graph as (*a*) becomes (*b*).



(*a*) (*b*)

Fig. 2

Omitting the 100 mark entirely misrepresents the information given by the original graph.

17·50. It should always be considered whether there is any meaning or value in drawing a curve through points plotted, and this affords an excellent exercise in common sense.

17·51. It may be pointed out that points are often, rightly or wrongly, joined by straight lines to enable the eye to survey the entire sequence of changes more swiftly.

17·6. Both with statistical graphs and continuous curves much oral questioning should be directed to bringing out a general idea of the information we can extract from a graph. Boys should be able to describe in simple terms what a graph tells them. They should be able to sketch on plain paper a rough graph from some such general information as 'the temperature rose steadily from 5 a.m. to 10 a.m., reached a maximum at 12 noon,' etc.

17·7. It is important from the outset to prevent the notion that a graph always starts from the origin.

17·8. It is essential that each graph be drawn or that a group of graphs be drawn with a definite aim in view. As far as possible, when a graph has been drawn several questions should be proposed about it. Graphs should have a definite heading, e.g. 'Graph to show . . . ,' or 'Graph to find . . . .'

17·9. Even with a graph of a continuous function it is frequently important that the points actually plotted should show up clearly on the final curve. They may be marked by a × or a ⊗.

§ 18. PROBLEMS IN ARITHMETIC *

18·0. Time is well spent in teaching problems even though there may appear to be little to show for it.

18·1. There is no way of teaching problems in general, so that all that can be done is to teach special problems. There is a danger in the over-classification of problems, such as has shown itself in the appearance in text-books of separate chapters with headings like Rule of Three, Work and Pipes, Alligation, Clocks, and Games of Skill, etc. The good teacher will organise his teaching with due regard to the way in which problems might be classified; he is bound to explain the types separately, but he will not, in general, set problems to his pupils as examples of special types. He will set them simply as problems to be solved by common-sense methods. The valuable sets of problems in a text-book are the miscellaneous sets.

18·2. It will sometimes be found better to set one problem to be solved in different ways rather than separate problems. For example, the use of algebra should not be forbidden, but there are certain problems that are better solved by arithmetic; if the pupil is made to do such a problem by both methods he can see for himself the advantage of the arithmetical solution.

18·3. Graphs are often useful but the pupil should not be allowed to depend wholly upon them.

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* See Ballard's *Teaching of the Essentials of Arithmetic* (Univ. of London Press).

§ 19. PERCENTAGE AND INTEREST

19·0. Taken literally, '4 % of x' means '4 for every 100 contained in x,' and its value is $4 \times \frac{x}{100}$.

It is also defined as meaning '4-hundredths of x,' which gives its value as $\frac{4}{100}$ of x.

19·1. A magnitude P is increased by 4 % by adding $\frac{4}{100}$ of P to P In simple cases of computation, e.g. in direct simple interest, this is the usual procedure. The same result is obtained by multiplying P by $\frac{104}{100}$. This is the usual procedure in profit and loss, in the general theory of compound interest, etc.

19·2. *Simple Interest.*

Both direct and inverse questions can be treated as cases of proportion based on the definition that r % of P means $r \times \frac{P}{100}$.

They can also be solved by a step-by-step argument, in which r % of P is taken as $\frac{r}{100}$ of P.

19·21. *Example* 1. Find the Simple Interest on £395 for 3 yrs. 4 mos. at $4\frac{1}{2}$ %. [Rough answer, £4 × 3 × 5 = £60.]

One year's interest is $\frac{4\frac{1}{2}}{100}$ of £395,

$3\frac{1}{3}$ years' ,, $\frac{4\frac{1}{2}}{100}$ of £395 $\times \frac{10}{3}$,

which comes to £59 5s. 0d.

19·22. *Example* 2. At what rate % will £360 amount to £420 in 3 yrs. 4 mos. ?

The principal earns £60 altogether ;

∴ ,, ,, £60 × $\frac{3}{10}$, or £18 in 1 year ;

∴ £100 ,, ,, £18 × $\frac{100}{360}$ in 1 year ;

the rate is 5 % p.a.

19·3. The inverse cases may also be dealt with by changing the argument of the equation $I = Prt/100$; but pupils should not use $P = \frac{100I}{rt}$ etc. unless they have made the transformation themselves.

To have done so is to have done a useful piece of algebra and to have obtained a quick and simple way of dealing with the inverse cases.

19·4. It may be pointed out that the examples given, like most examples in the text-books, are artificial. In practice, simple interest on loans is only paid for short periods; thus in banks where accounts are balanced and accrued interest added in each quarter, or each half-year, the interest becomes compound. But the examples provide easy exercises introductory to the more difficult cases of (1) Bank interest for a period of days, and (2) Bill-discounting.

19·5. *Compound Interest.*

Compound interest operates when the account is made up at regular intervals in such a way that the interest as it becomes due is added to the principal. This is the case in the calculation of mortgages, annuities and life insurances. In the actuarial theory, money invested is regarded as being increased continuously in such a way as to result in a certain annual percentage increment.

19·51. In such a sum as 'Find the compound interest on £395 in $3\frac{1}{2}$ yrs. at 4 % p.a.' it is usually assumed in the text-book that the interest is added annually for 3 yrs., and half a year's interest (i.e. 2 %) is added for the last half-year. In practice, if the account is made up annually, interest might not be earned for the last half-year; if the account is made up half-yearly, by the addition of 2%, the effective annual rate would be 4·04 %, so that to add 2 % in the case quoted would be rather too much for a half-year's interest. In the same way, 1 % per quarter would mean an effective annual rate of 4·060401 %.

The general formula, which is consistent with the theory of continuous growth, would give the amount as $£395 \times 1{\cdot}04^3 \times \sqrt{1{\cdot}04}$.

19·52. Questions in compound interest provide exercise in:

(1) straightforward computation, using a compact arrangement of working;

(2) logarithms, using the formula $A = PR^t$, where $R \equiv 1 + \frac{r}{100}$;

(3) making and using tables, including those used for annuities, etc.;

(4) algebraical manipulation of the formula, introducing a new type of equation when t is the unknown.

19·6. *Notes.*

(1) All questions of simple interest, direct and inverse, are questions of proportion, the interest—but not the amount—being terms of the proportion. But in questions of compound interest

only the principal and the amount are in proportion. Thus in S.I. the interest, in C.I. the amount, is the important thing.

(2) For making rough approximations, it may be taken that C.I. is slightly greater than the S.I. for the same rate and time, if the time is short; but for long periods, if the S.I. is 70 % of the principal, the amount at C.I. is about double the principal; if the S.I. is 35 % of the principal, the amount at C.I. is about 1·41 times the principal.

(3) Some teachers prefer to do simple cases of C.I. before introducing S.I.

(4) The amount of £1 at a nominal rate of 100 % C.I. p.a. payable as 100 % yearly, 50 % half-yearly, 25 % quarterly, etc., provides an arithmetical introduction to *e*.

§ 20. DISCOUNT

20·1. A generation or more ago 'True Discount and Present Worth' was a regular element in school arithmetic. Then scorn was poured upon it on the ground that it was not 'practical' and that business men did not use it. Being perhaps unpopular because it was difficult, it was suppressed.

20·2. This surrender may or may not have been wise; whether it was or not, the train of thought that lies behind the subject cannot be dispensed with if arithmetic is to be properly done, for it was a clear and typical case of the 'inverse process.' Nor is it true that 'present worth,' in the strict sense, is an unpractical idea; it is in constant use. The only 'unpractical' element in the old discount sums was their application to the discounting of *bills* with short periods—months, not years—to run.

20·3. The essence of the contrast between the 'direct' and the 'inverse' process as here exemplified needs to be consciously and clearly seized. In the direct question, the quantity to be operated on is known at the outset, e.g. find the interest on £520; or, a house that cost £1200 is sold at a profit of 30 %, etc. In the inverse process, the quantity on which operation must begin is not known or has to be assumed, e.g. as x or as 100 or as 1. Or to state the contrast otherwise: in some questions all the materials for the top line of a reduction to unity sum appear in the question in the form wanted: in the inverse sums they do not. They have to be manufactured from the materials given, and then the top line so obtained has to be turned round or restated so that the quantity sought comes at the end.

20·4. The type of question fashionable to-day in which the difficulty appears most sharply is 'profit and loss'—and it is whispered that some want to get rid of, i.e. to slur over, the difficulty here, a policy with which the Committee emphatically disagrees. For if the difficulty is ignored in one place it turns up in another and is in fact ineluctable.

20·5. Business men have their own well-established and well-understood customs; everyone knows that if a banker is ready to discount a bill of 5½ % the percentage is reckoned on the face value of the bill; that if a shop offers a reduction of 3d. in the shilling this means on the nominal price. But if a business man says to himself 'I have lost half my capital in this year's trading,' he is surely thinking in terms of his original capital. If, on the other hand, he says 'My receipts have been . . . and 10 % of that was clear profit,' he is thinking in terms of the receipts, not of the capital.

20·6. If it is said that the population of a town has increased 20 % in the last 10 years, the meaning is plain to all educated people, and the two types of question arise according as the initial or the final population is given.

20·7. As regards school work we aim at cultivating clear thinking as well as facility of calculation, and an essential part of dealing with such questions is to learn to recognise their exact meaning ; whether the business world uses its terms in their strictly proper sense or not, our business for the time being, i.e. for school work, is to do so.

20·8. If some business terms and customs are of sufficient importance to make special reference to them desirable, that should be done in addition and any variance between them and the strict natural interpretation pointed out, but they should not be allowed to replace or obscure the scientific use.

§ 21. RATE OF PROFIT OR LOSS

21·1. As noted in the previous section on 'Discount,' questions as to rate of profit and loss fall into two classes, the easy direct case, and the more difficult inverse case, corresponding exactly to 'Interest' and 'Discount.' Hence in any such question the essential points are :

(1) on what quantity is the rate to be reckoned ?

(2) is it the easy direct case or the difficult inverse case ?

Unless a boy is quite clear as to these two points before he begins calculating, he is little likely to keep straight.

21·2. A few notes on minor points may be added :

(1) 'On' is a dangerous word with a double meaning, and it may be better to say, e.g. 'By spending £8 you gain 2s.' than 'On £8 you gain 2s.'

(2) In the direct case it is often better to calculate the gain or loss and add or subtract than to use the factor $1 \pm \frac{r}{100}$. E.g. 'a pig was bought for £7 18s. 4d. and sold at a loss of 3 % ; what was it sold for ?' Better calculate 3 % of £7 18s. 4d. and subtract than calculate 97 %.

(3) For the inverse case there are three reasonably good ways of proceeding :

E.g. By selling sugar at $4\frac{1}{2}$d. a lb. a grocer gained 26 %. What did a hundredweight cost him ?

He sold 1 cwt. for $\frac{3}{8} \times 112$ sh., and

He bought a cwt. for x sh. ;

$\therefore x + \frac{26}{100}x = \frac{3}{8} \times 112$, etc.

[The danger here is the omission of the second x.]

(*b*) What he sold for 126 he bought for 100, etc.

(*c*) He gains 26 % ;

$\therefore$ Selling price $= \frac{126}{100}$ of cost price,

and Cost price $= \frac{100}{126}$ of selling price, etc.

(4) In the long run, familiarity in forming the necessary multiplier, exemplified in (*c*) above, is important, and it

may perhaps be said that not till this has been acquired is the difficulty fully mastered. Such facility becomes especially useful when the transaction contains several steps, and the final operator will consist of several such factors.

(See Section (·2) of the Appendix.)

§ 22. SHARES AND STOCKS

22·1. The actual arithmetical computation involved in this subject is usually simple, but the conception of the commercial machinery involved is, for a boy, unusually difficult. This is probably due to the fact that shares and stocks rarely come within the everyday experience of the average boy and naturally he finds it difficult to comprehend ; to him it is practically an abstract idea. It is a subject suitable for senior boys only, for here is prospect of success and reasonable hope of reward for the usually lengthy time expended upon it.

22·2. Probably the best approach is by means of preliminary chats on the necessity for shares and stocks. To expect a boy to be cognisant of the many different kinds of shares, or some of the practical terms involved in buying and selling is rather to cause confusion than disperse it. He must possess, however, some elementary knowledge of the meaning of capital, profits, dividends ; prices and terms relative to them ; procedure employed in buying and selling stock. He must realise that shares and stocks are things which can be bought and sold like any other commodity, although the method of selling is not the one with which he is usually familiar. It will be of interest to discuss with the class the factors upon which the value of a stock depends—supply and demand, success of the company, object of loan being raised, future prospect of company, etc. A perusal of the daily shares and stocks lists in the various newspapers may be of interest, and suggestions as to possible reasons of fluctuations in prices will often be forthcoming. It is a matter of surprise to most boys that all the investor gets to show his purchase is a certificate or voucher recording his investment. A boy will readily begin to see that investing in shares and stocks may be somewhat of a gamble, unless the buyer is merely making an investment to secure a regular income, and it will be obvious that a high rate of interest suggests a corresponding risk. It is worthy of note that at the time of writing the average yield is slightly under five per cent. (written in 1931).

22·3. It is suggested that the question of shares should receive first consideration, as the treatment is slightly simpler than that of stocks. From the first the boy must be perfectly conversant with the exact meaning of such a phrase as 5 % Great Western stock at 93. If he really understands that this means :

(1) £100 Stock can be bought for £93 cash,

(2) £100 Stock yields £5 Cash yearly,

(3) £93 Cash invested yields £5 Cash yearly,

then he is well equipped for his battle with this troublesome subject.

22·31. It is frequently found that boys get confused between stock and cash, and it is recommended that the word stock is inserted where it is implied, and cash where money is involved, as above.

22·32 Examples should be simple and direct at the commencement of the study and be limited to definite types, such as:

(1) Given a certain amount of stock, find its value according to data.
(2) Given a sum of money, how much of a certain stock may be bought.
(3) Find income from given investment.
(4) Find income from given holding.
(5) Find return per cent. on money invested.

22·4. The charges incurred in purchasing and disposing of shares and stocks will have to be considered, e.g. transfer fees, stamp duties, brokerage, etc., and will be introduced at the teacher's discretion or as exercises demand. Brokerage is always a trouble to a boy and some examining authorities omit it on this account. It should be noted that the charge involved in brokerage is a variable quantity and not always that mysterious $\frac{1}{8}$ %, so beloved of authors of text-books of a previous generation. The stockbroker naturally must charge for his services in a purchase or sale of stock, and therefore when buying stock the brokerage becomes a charge on the price and hence is added to the price of the stock, while when selling stock the brokerage is a deduction from the price of the stock, since the broker naturally deducts his charges from what he gets from his sale. The deduction of income tax offers another complication, but the boy should endeavour to see that this implies that the stockholder receives only a certain fraction of the income the stock yields. For example, if income tax is 4s. 6d. in the £, then the shareholder receives only $\frac{31}{40}$ of his yield, the remaining being taken as income tax.

22·5. The involved question in shares and stocks can only be solved by a boy who possesses a thorough appreciation of the initial stages of the subject, and it is advisable to give a thorough grounding in these before proceeding to the more complex examples.

§ 23. APPROXIMATIONS

23·1. The important thing about approximate answers is that they should be right *as far as they go*. If we only know that Jones lives in Hampshire, we can begin to locate him with 'England, South Coast, Hampshire, . . .,' but must not continue with 'Bournemouth'; each new piece of information improves our knowledge of his residence until we come to the last, and that will be wrong if, for instance, he lives in Portsmouth.

23·2. Many boys (and others) think that increased accuracy can be obtained by carrying a calculation to more figures, and that it is virtuous to give greater accuracy than that which is asked for. But when the data are only approximate, the extra figures are often wrong and spoil the answer. If Jones' room in Portsmouth is marked approximately by a point on a map of the world, successive magnifications of the map may show the point in a district, street, and house in Bournemouth with an appearance of accuracy which is really quite fallacious.

23·3. No practical measurements can in fact be known to be exact; the degree of accuracy may be specified in different ways. Thus ·370 metres gives a length correct to three significant figures if it means that the length is 370 mm. to the nearest mm., and this implies that the length lies between 369·5 and 370·5 mm. The error is thus not more than ·5 mm. The measurement is sometimes written $(370 \pm \cdot 5)$ mm.

23·4. Significant figures may be defined as those which must be retained for any position of the decimal point; e.g. ·00602 kilometres = 6·02 metres = 6020 millimetres, where neither the first zeros in the first stage nor the last zero in the last stage is the figure which essentially remains, whereas the middle zero does. Thus it is essential to work according to the number of significant figures given or required rather than to use the number of decimal places which alters, except in money sums in which it is usual to work to so many decimal places of a £.

23·5. Even in 'shop' arithmetic, approximation is inevitable. The butcher charges 5s. 5½d. for 3 lb. 7 ozs. of meat at 1s. 7d. a pound since he cannot charge 5s. $5\frac{5}{16}$d. The pupil might give the answer as:

arithmetic answer, 5s. $5\frac{5}{16}$d.;
shop answer, 5s. 5½d.

If he only gives 5s. 5½d. it should be accepted.

23·6. Consider the example: A rectangular block is measured as 3·27 in. by 2·63 in., by 1·95 in.; find its volume.

The answer would be 16·770195 cu. in., only if the data were exact, but there is no sense in giving the answer correct to the millionth of a cubic inch.

23·61. Actually such measurements would be good if subject to no greater liability of error than $\frac{1}{100}$th of an inch in each dimension, and then the volume lies between

$$3{\cdot}26 \times 2{\cdot}62 \times 1{\cdot}94 = 16{\cdot}5699\ldots \text{ cu. in.}$$
$$3{\cdot}28 \times 2{\cdot}64 \times 1{\cdot}96 = 16{\cdot}9710\ldots \text{ cu. in.}$$

23·62. Or, taking the measurements to possess three-figure correctness, the volume lies between

$$3{\cdot}265 \times 2{\cdot}625 \times 1{\cdot}945 \simeq 16{\cdot}670 \text{ cu. in.}$$

and

$$3{\cdot}275 \times 2{\cdot}635 \times 1{\cdot}955 \simeq 16{\cdot}871 \text{ cu. in.}$$

The answer should be given as 'about 16·8 cu. in.' on the general principle of answering to as many significant figures as the data, but boys should be taught at an early stage that the third figure is unreliable. It is instructive to work out the product $3{\cdot}27 \times 2{\cdot}63 \times 1{\cdot}95$, marking every figure that would be affected by the introduction of a fourth significant figure.

23·63. Experience will show, as in the above example, that in general with n-figure data the answer is nearly correct but not reliably correct to n-figures, and that an n-figure answer is a mean of the extreme possibilities.

Many rules for deciding how many figures are trustworthy are based on probability. For example, $3\frac{1}{7}$ for π gives the third figure right more than 5 times out of 6, but the fourth figure wrong more than 6 times out of 7. Beginners should be protected, by careful choice of questions, against having failures of these rules forced upon their attention until their faith in the rules is strong enough to stand the strain. No approximation is accurate enough to give an answer correct to even one figure in every possible case. The loss of an eyelash may shift your centre of gravity from England into Scotland.

23·71. To say, in approximate work, that an angle is 17·50, 17·5, and $17\frac{1}{2}$ degrees does not mean the same thing. The first statement, by convention, implies 4-figure accuracy—a result between 17·495 and 17·505; the second, 3-figure accuracy—a result between 17·45 and 17·55. The third statement is not so clear in its implication; it means we cannot tell whether the angle is nearer 17 or 18; it might mean a result between $17\frac{1}{4}$ and $17\frac{3}{4}$, especially if given as an answer to a measurement.

23·72. The distinction between 17·50 and 17·5 can scarcely be avoided, and may be compared with the statement that when we say the distance of the sun is 93000000 miles 'all our noughts should be crosses,' but the average boy should not be worried by the distinction between ·5 and $\frac{1}{2}$.

23·8. A bad way of approximating to a small quantity is to find (approximately) two large quantities of which it is the difference; the weight of the captain's watch is not well found by estimating the ship's displacement before and after it fell overboard.

23·81. In finding $\frac{9}{80}$ of £17 2s. 11d., the multiplication should be done before the division because of the multiplication of the approximation-error in the opposite process.

23·82. To evaluate $\frac{\sqrt{3}}{3}$, given $\sqrt{3}=1·7321\ldots$.

Here it is evident that an answer to 4 places of decimals is wanted, no more being obtainable from the data.

$\frac{1·7321}{3}=·577366\ldots$ should be avoided, since it is based on the assumption that $\sqrt{3}=1·732100\ldots$.

The correct procedure is to go straight from $\frac{1·7321}{3}$ to ·5774, obtaining the last figure 4 from the division of 11 by 3 and arguing that $11\div 3$ is nearer to 4 than to 3, because 11 is nearer to 12 than to 9.

23·83. The sign meaning 'is approximately equal to' need not be used to excess; e.g. in $\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}\doteqdot\frac{1·7321}{3}=·5774$, the first sign must be $=$, the second $\doteqdot$; strictly the third ought to be $\doteqdot$, but it may be replaced by $=$, with the words 'to four places' understood.

23·84. One of the worst faults is to substitute approximations too soon, especially in expressions like $\frac{4\pi}{3}+\frac{2\pi}{9}$. This is an offence against the rule, 'never do a piece of work until you are obliged to do it.'

23·85. Ideas on approximation tend more than those on the other subjects to be vitiated by slovenly thinking. The superstition that if nothing is perfect one thing is as good as another, is not confined to the young. Mathematics ought to make a better contribution than this to human life, which should be founded on the conception that we should all be genuine as far as we go, and the best available approximation to something better.

23·86. Cases will arise in which the best answer that a boy can get is not expressible in any regularly-used unit, such as 'nearest penny,' '4 significant figures,' or 'nearest minute.'

In such cases it may be necessary to give the answer in such a form as '£4 9s. 4d. within a penny' (if half the answer has been found to be £2 4s. 8d. to the nearest penny).

If 4-figure logarithms are used to multiply 4·473 by 167·1, the logarithm of the answer is 2·8736, and the nearest approach to a four-figured answer which can be deduced from this is '747·5 within ·1.' Or if the angles A and B of a triangle are 43° 13′ and 77° 20′ (to the nearest minute), then $C = 59° 27'$ within a minute, but not necessarily to the nearest minute. On this plan the volume in section 23·6 might have been given as '16·77 cubic inches, within ·10.'

This should not be the regular form for answers, but should be accepted when circumstances prevent the use of the ordinary forms.

§ 24. FURTHER GENERAL REMARKS ON THE SCOPE OF THE REPORT

24·1. So far, this report has dealt with an arithmetical course which may be regarded as a minimum for all pupils ; a minimum, that is, whether the ultimate object to be gained is the acquisition of sufficient facility in using arithmetical processes in the problems of daily life, or merely the passing of some first school examination, to say nothing of the cultural value of the study.

24·2. After this minimum course has been concluded, mathematical periods will probably be used for the development of other branches of mathematics, but in some schools it may be convenient or desirable to devote more time to arithmetic, e.g. in commercial schools time will naturally be given to business arithmetic, book-keeping, etc. The later sections of this report deal with possible methods of employing this time, if it is available, but none of them need be regarded as indispensable items in the curriculum of the average school.

24·3. We do not advocate for every boy a separate course on the 'Arithmetic of Citizenship.' The mathematical principles involved will be obscured by unfamiliar terminology. Interest in such subjects as stocks and shares, taxation, banking and insurance, is an interest that develops naturally at a comparatively late stage in school life ; it is a matter of age rather than mental development ; and the mathematics involved in text-books on 'Citizenship' should not in itself present any special difficulty. This does not mean that the teaching of the lessons in 'Citizenship' is unimportant ; it only means that it is not mathematics, and need not necessarily be the concern of the mathematical teacher.

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§ 25. ARITHMETIC OF CITIZENSHIP

25·1. The arithmetic of citizenship may be taken as a school study by two main types of pupils :

(*a*) The pupil who has an extra year of his school life to ' live ' after he has taken his first school examination and who is not likely to profit by proceeding to the more advanced branches of mathematics.

(*b*) The pupil who wishes to embark upon a commercial career.

25·2. The outline of the course suggested below and the detail is sufficient to indicate the topics discussed.

The main headings of the syllabus are :

(1) Local and County Finance.
(2) National Finance.
(3) Saving, Banking, Investing (from the Investor's point of view).
(4) Capital and Industry.
(5) Insurance.
(6) Compound Interest in Finance. (This should be discussed without special reference to algebraical formulae and without the study of progressions.)

25·3. It is usually found that such matters as shares and stocks, banking, simple and compound interest, introduced as essential parts of the above course, create an interest in arithmetic which no other type of syllabus even suggests. In schools where there is a commercial course as part of the curriculum the work of correlation of subjects is easy. The commercial master will emphasise in his book-keeping and economics course the kind of problem to be solved, and it will be left for the mathematics master to deal with its solution. The circumstances of the moment will determine the kind of problem to be tackled. Current events, both locally and nationally, will decide to a large extent when certain branches of the work are to be considered, e.g. The Budget.

25·4. The introduction of graphs into commerce may very well have as their starting point a careful study of the cost of living index figures published monthly by the Board of Trade. Boys in the ' woollen areas ' will have practice in making their own ' wool price charts.'

25·5. A fund of statistical information can be obtained from such publications as :

(1) Whitaker's *Almanack.*
(2) *The Statesman's Year Book.*
(3) Parts of *Encyclopaedia Britannica.*
(4) The daily newspaper.
(5) Government returns and publications.
(6) Insurance tables.

25·6. The Schools Savings Association will be found to be very useful as an illustration of corporate savings. All pupils should understand how an income-tax form is 'filled up,' and should also appreciate the meaning and use of 'abatements.' Further details on the subject may be found in :

(1) Appendix A ; *Girls' Schools' Report*, pages 41, 42, 43.
(2) Mr. Gimson's Paper, *Mathematical Gazette*, March 1930, pages 55-61.

§ 26. BUSINESS ARITHMETIC AND ACCOUNTS

26·1. The arithmetical rules and processes needed in the practice of double entry book-keeping are, in the main, those with which the pupils of secondary schools are familiar. The more definite vocational aim and treatment, the special applications and the cultivation of economy of time are, however, new features.

26·2. The first things to be understood in book-keeping are the two-fold nature of the transaction: the advantage of the double entry—particularly as a check upon entries; the trial balance—what it does, and does not reveal.

26·3. Facility in the use of simple processes—e.g., rapid decimalisation of money, the use of decimals in cost calculations, interest, percentages—must be acquired as soon as possible, and an early introduction should be given to 'short methods.' The early teaching of book-keeping will centre round simple ledger accounts (including a cash account), the purchase and sale of goods and the results of a period of trading (sole trader). The corresponding arithmetical work may be:

(1) Invoices, debit and credit notes, discounts (trade and cash—cash discount being shown to be of the nature of interest), gross and net prices—with the calculation of one from the other.

Such calculations will serve as an introduction to purchases, sales and return books, as books of original entry.

(2) Cost calculations; introduction of the cash book—with transfers from bank to office cash and *vice versa*, recording cash transactions.

(3) Methods of payment—the cheque system and the effecting of payments by means of book-keeping entries (bank to customer, customer to bank, etc.).

(4) The results of a trading period—simple trading and profit and loss accounts.

Calculations:

(*a*) Gross profit per unit sold; percentage on cost and selling prices.

(*b*) Gross profit for a period; percentage on turnover (net sales).

(*c*) Net profit for a period; percentage on turnover and on capital.

(*d*) Working expenses; relations to turnover, net profits.

(*e*) Graphs and graphical illustrations.

The interpretation of accounts, especially as regards the above calculations (*a*) to (*e*) and the effect upon business policy is particularly important. The trading and profit and loss accounts are particularly useful as bases for this work.

(5) Interest—ordinary calculations, the third, tenth and tenth rule, calculation of interest on deposit accounts, loans, etc., true discount, compilation and use of interest tables.

(6) The exchanges—conversion of currencies, commercial (i.e. banker's) discount, present worth, bills receivable and bills payable. (*N.B.*—The bill of exchange as a special application of interest.)

(7) The arithmetic of partnerships and joint-stock companies—sharing profits (proportionally to capital or otherwise), the issue of stock and shares, sinking funds, depreciation, changes in capital, Stock Exchange transactions, dissolution of partnership, winding up of companies.*

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* Book of reference: *Commerce, An Introductory Study* (Oxford Univ. Press).

§ 27. STATISTICS

27·1. It is one of the outstanding defects of our mathematical education that in the great majority of courses it fails to include even the most elementary knowledge as to the handling and interpretation of statistics. There are large classes of men who have to deal with figures in the course of their professions: for example, actuaries, accountants, social workers, politicians, and many business men. For such men knowledge of statistics is likely to be of great value.

27·2. Even the knowledge that the *handling* of statistics needs the use of a special and definite technique, if the results are not to be seriously misleading, is of itself of value. *If* they are properly handled it *is* possible to prevent them from being the superlative of which lies are the positive.

27·3. To advance far into the study of statistical method needs a considerable mathematical equipment, including familiarity with the calculus, but the short course outlined below can be tackled by anyone who is capable of reaching matriculation standard. It could be worked through in one term in two or three hours weekly, according to the ability of the worker and the number of examples attempted.

27·4. *Syllabus of Short Course in Statistics.*

FREQUENCY: Histograms, frequency curves. How the accuracy of measurement affects the class-interval.

AVERAGES: The mean, median, and mode.
Calculation of the mean, using an arbitrary origin.
Calculation of the median.

DISPERSION: Standard deviation. Interquartile range.

CORRELATION: The two lines of means (lines of regression).
Pearson's coefficient of correlation, found as the square-root of the ratio of the gradients of these two lines. (The calculation of this coefficient by the usual method presents some difficulty.)
Spearman's coefficient of rank correlation—for comparing two orders.

PROBABILITY : Some preliminary work on combinations.
The binomial graph, produced by the terms of $(\frac{1}{2}+\frac{1}{2})^n$.
The curve of normal probability.

SAMPLING : Some idea of its probable error.

Reference to the following books is suggested :

A First Course in Statistics, D. Caradog-Jones. (Bell.) 15s.—has very few examples.

Introduction to the Theory of Statistics, G. Udny Yule. (Griffin.) 12s. 6d.—some examples.

Secondary School Examination Statistics, J. M. Crofts and D. Caradog-Jones. (Longmans.) 2s. 6d.—introduces the ideas shortly and clearly ; no examples.

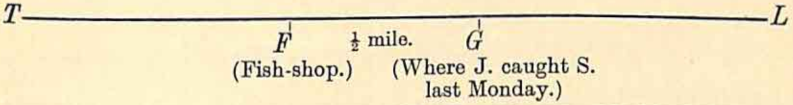
Elementary Manual of Statistics, A. L. Bowley. (King.) 7s. 6d.—there is a key to the examples, published at 6s.

Examples will be found in *Exercises in Algebra*, Pt. II, T. P. Nunn. (Longmans.) 7s. 6d.

§ 28. APPENDIX OF WORKED EXAMPLES

(·1) *Arrangement of Work by Tabulation.*

Jones, who bicycles at 12 m.p.h. from Tooting to London, regularly overtakes Smith, who walks at 4 m.p.h., opposite the same fish-shop. Last Monday Jones started late, and overtook Smith ½ mile after the fish-shop. If Smith was punctual, how late was Jones ?



	Ordinary day.		Last Monday.	
	F	*G*	*F*	*G*
Jones passed -	7 : 30 *A*	7 : 32½ *D*	7 : 35 *H*	7 : 37½ *G*
Smith passed -	7 : 30 *B*	7 : 37½ *C*	7 : 30 *E*	7 : 37½ *F*

(The letters show the order in which the rabbit-hutches are filled up. The 7 : 30 in *A* is purely arbitrary.)

∴ Jones was 5 min. late.

(·2) *Use of Multiplying Factor in Percentages.*

Example 1. The population of a town increases by 10 % and 15 % in successive years. How much % must it decrease in the 3rd year to return to its original number ? [Rough answer about (10 + 15) %.]

Let P, P_1 and P_2 be the population originally and after the 1st and 2nd years, then :

$$P_1 = P \times \frac{110}{100},$$

$$P_2 = P_1 \times \frac{115}{100} = P \times \frac{110}{100} \times \frac{115}{100}.$$

To return to P, P_2 must be multiplied by $\frac{100}{110} \times \frac{100}{115}$

$$\text{i.e. by } \frac{200}{253}, \text{ which } \simeq \cdot 7903$$

$$\text{i.e. } \frac{79 \cdot 03}{100}.$$

I.e. it decreases by nearly 21 %.

Example 2. At the end of a year the daily circulation of a paper was 15 % greater than at the beginning. In the 2nd and 3rd years the corresponding increases were 10 % and 8 %. Find the percentage increase in the whole period of 3 yrs.

[Rough answer about 35 %.]

Let C, C_1, C_2, C_3 be the original circulation, and the circulation after the 1st, 2nd and 3rd years.

Then

$$C_1 = C \times \frac{115}{100},$$

$$C_2 = C_1 \times \frac{110}{100} = C \times \frac{115}{100} \times \frac{110}{100},$$

$$C_3 = C_2 \times \frac{108}{100} = C \times \frac{115}{100} \times \frac{110}{100} \times \frac{108}{100}.$$

[After a little practice the pupil should write the last line straight off.]

This reduces to $C_3 = C \times \frac{136{\cdot}62}{100}$.

The required percentage is 37 % (to the nearest 1 %).

(·3) *Compound Interest.*

A borrows £1000 at 6 % C.I., and repays £280 at the end of every year. How long will it take him to clear off the debt ? How much is his last payment ?

$$\begin{array}{rl}
\pounds 1000 & P_1 \\
60 & I_1 \\
-280 & R_1 \\
\hline
780 & P_2 \\
46{\cdot}8 & I_2 \\
-280 & R_2 \\
\hline
546{\cdot}8 & P_3 \\
32{\cdot}808 & I_3 \\
-280 & R_3 \\
\hline
299{\cdot}608 & P_4 \\
17{\cdot}97648 & I_4 \\
-280 & R_4 \\
\hline
37{\cdot}5845 & P_5 \\
2{\cdot}255070 & I_5 \\
\hline
\pounds 39{\cdot}8396 & R_5 \\
20 & \\
\hline
16{\cdot}792 \text{ s.} & \\
12 & \\
\hline
9{\cdot}504 \text{ d.} & \\
\hline
\end{array}$$

The debt is paid off in 5 years.
The last payment is £39 16s. 10d. } *Ans.*

Note.—This should appeal to an active-minded boy as a welcome variation on the ordinary form of Compound Interest question. Climbing up at an increasing rate out of a slough of debt is more stimulating than mere accumulation of unearned wealth.

(·4) *Change of Units.*

In questions involving a number of transformation of units, too often the pupil scatters the data into a number of small sums, the results of which he endeavours to combine; too often he finds himself plunged into uncertainty and confusion. The following examples show how he can avoid this by using an argument directly developed and clearly stated.

He must first ask himself what is the unknown.

Thus in the example: Assuming 1 m. $=39{\cdot}37''$, 1 gallon $=277{\cdot}27$ cu. in., find correct to 3 figures the number of pints in 1 litre, the unknown is the English equivalent to a metric measure of capacity. Next, the pupil looks for a statement connecting English and metric measures and writes it down with the English at the end.

$$1 \text{ metre} = 39{\cdot}37''.$$

This is linear measure; he proceeds to cubic measure:

$$1 \text{ cu. m. or } 1000 \text{ cu. dm.} = 39{\cdot}37^3 \text{ cu. in.}$$

He now transforms to measures of capacity:

$$1000 \text{ litres} = \frac{39{\cdot}37^3}{277{\cdot}27} \text{ gall.}$$

$$= \frac{39{\cdot}37^3 \times 8}{277{\cdot}27} \text{ pints.}$$

$$1 \text{ litre} = \frac{39{\cdot}37^3 \times 8}{277{\cdot}27 \times 1000} \text{ pints.}$$

This works out to 1·76 pint.

Rough answer gives $\frac{40^3 \times 8}{280 \times 1000}$, i.e. $\frac{64 \times 2}{70}$ or 1·8 pints, but the pupil probably knew the correct answer before doing the sum. It is useful in drill work of this kind to set sometimes a question of which the pupil knows the answer beforehand.

An alternative method of solution is :

We have to connect litres and pints ;

$$\begin{aligned} \therefore \text{ we say } 1 \text{ litre} &= 1000 \text{ c.c.},\\ (100)^3 \text{ c.c.} &= 1 \text{ cu. metre},\\ 1 \text{ cu. metre} &= (39{\cdot}37)^3 \text{ cu. in.},\\ 277{\cdot}27 \text{ cu. in.} &= 1 \text{ gall.},\\ 1 \text{ gall.} &= 8 \text{ pints},\\ \therefore\ 1 \text{ litre} &= \frac{8 \times (39{\cdot}37)^3 \times 1000}{(100)^3 \times 277{\cdot}27} \text{ pints}. \end{aligned}$$

[This is an example of what used to be known as chain rule, a method particularly useful in exchange questions.]

An example illustrating the first method and the method of the examples in Section (·2) of this Appendix is :

In France the 2nd class railway fare is 48 centimes per kilometre. Taking 1 metre as 39·37″ and £1 as 124·3 francs, find by how much per cent. the French rate differs from the English rate of $1\frac{1}{2}$d. per mile.

[Rough answer—In France, $\frac{5}{8}$ mile costs less than 1d. ; i.e. 1 mile costs less than 1·6d. Thus a rough answer for the difference only tells that it is small.]

$$1 \text{ kilometre costs } 0{\cdot}48 \text{ francs};$$

$$\text{i.e. } 39370'' \text{ cost } £\frac{48}{12430}$$

and $$1 \text{ mile or } 63360'' \text{ cost } \frac{48}{12430} \times 240 \times \frac{63360}{39{\cdot}370} \text{ pence},$$

which works out to 1·4915 pence.

The ratio of the French to the English price is

$$1{\cdot}4915 \times \frac{2}{3}, \ \text{ i.e. } {\cdot}9943 = \frac{99{\cdot}43}{100}.$$

The French price is 0·57 % cheaper (within an error of 0·01 %).

(·5) *Numerical Evaluation of Series.*

1. Evaluate the series $4\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\dots\right)$.

(1) taking x as $\frac{1}{2}$, (2) taking x as $\frac{1}{3}$.

Give the sum of the two results, correct to 4 places of decimals. Do you recognize this sum ?

2) 4·00000	Plus		Minus
4) 2·00000 ÷ 1 =	2·00000		
4) ·50000 ÷ 3 =			·16667
4) ·12500 ÷ 5 =	·02500		
4) ·03125 ÷ 7 =			·00446
4) ·00781 ÷ 9 =	·00087		
4) ·00195 ÷ 11 =			·00018
4) ·00049 ÷ 13 =	·00004		
4) ·00012 ÷ 15 =			·00001
4) ·00003	2·02591	–	·17132
·00001	– ·17132		
	1·85459 = Sum when $x=\frac{1}{2}$.		

3) 4·00000	Plus		Minus
9) 1·33333 ÷ 1 =	1·33333		
9) ·14815 ÷ 3 =			·04938
9) ·01646 ÷ 5 =	·00329		
9) ·00183 ÷ 7 =			·00026
9) ·00020 ÷ 9 =	·00002		
·00002	1·33664	–	·04964
	– ·04964		
	1·28700 = Sum when $x=\frac{1}{3}$		
	Add 1·85459 = Sum when $x=\frac{1}{2}$		
	3·14159		

Sum of the two results, correct to 4 places, $=3·1416$.
This looks like π. } *Answer.*

Note.—With more experience, many of the 0's may be omitted. This abbreviation is used in the next example.

2. In the series $1+\frac{x}{1}+\frac{x^2}{1\times 2}+\frac{x^3}{1\times 2\times 3}+\frac{x^4}{1\times 2\times 3\times 4}+\dots$ take x first as $\frac{1}{3}$ and then as $-\frac{1}{3}$. Evaluate to 8 significant figures in each case. Look for some connection between the two answers.

(Successive terms are got by dividing the preceding term by 3, 6, 9, 12, etc.)

Plus in both cases		Plus if $x=\frac{1}{3}$ Minus if $x=-\frac{1}{3}$
1·000 000 000		·333 333 333
55 555 556		6 172 840
514 403		34 294
1 905		91
4		
1·056 071 868	±	·339 540 558
± ·339 540 558		

$$\left.\begin{array}{ll} 1{\cdot}395\ 612\ 4 & = \text{sum when } x=\ \ \frac{1}{3} \\ {\cdot}716\ 531\ 31 & = \text{sum when } x=-\frac{1}{3} \end{array}\right\} \textit{Answer.}$$

$$\begin{array}{r} {\cdot}71653131\dots \\ 1{\cdot}395612426\ \overline{)\ 1{\cdot}0000000000} \\ 9769286982 \\ \hline 230713018 \\ 139561243 \\ \hline 91151775 \\ 83736746 \\ \hline 7415029 \\ 6978062 \\ \hline 436967 \\ 418684 \\ \hline 18283 \\ 13956 \\ \hline 4327 \\ 4187 \\ \hline 140 \end{array}$$

The product of the two sums is 1.

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